WARSASH MARITIME SCHOOL

PART OF SOLENT UNIVERSITY SOUTHAMPTON

Why do marine engineers need to study mathematics?

Marine and electro-technical engineering cadets at Warsash are required to study a variety of subjects prior to going to sea. These will include marine engineering principles and electrics before moving on to magnetism and, finally, mechanics which considers how both static (still forces) and dynamic forces (moving forces) act on an item of equipment.

In applied heat, electrics, and mechanics you will need to demonstrate your understanding of new ideas and concepts through answering descriptive questions and questions that require you to calculate numerical answers. The mathematics that you study during your first academic session as an engineering cadet will help you to develop the mathematical skills to enable you to tackle these numerical questions.

As you have probably recognised, it is important that you have mastered certain mathematical skills early in the course. Below you will find also a collection of mathematical notes from www.mathcentre.ac.uk to help you in your preparation for your cadetship.

This website has been developed to support students in mathematics at a variety of levels. It includes notes, videos, self tests etc and will enable you to develop the prerequisite mathematical skills for your marine engineering cadetship. We strongly recommend that you take some time prior to joining us at Warsash to develop these skills.

Contents

Mathematical language

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This introductory section provides useful background material on the importance of symbols in mathematical work. It describes conventions used by mathematicians, engineers, and scientists.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- understand how important it is to be precise about the symbols you use
- understand the importance of context when trying to understand mathematical symbols

Contents

1. Introduction

Mathematics has its own language, much of which we are already familiar with. For example the digits

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

are part of our everyday lives. Whether we refer to 0 as 'zero', 'nothing', 'nought', or 'O' as in a telephone number, we understand its meaning.

There are many symbols in mathematics and most are used as a precise form of shorthand. We need to be confident when using these symbols, and to gain that confidence we need to understand their meaning. To understand their meaning there are two things to help us -

context - this is the context in which we are working, or the particular topics being studied,

and

convention - where mathematicians and scientists have decided that particular symbols will have particular meaning.

2. Some common mathematical symbols

Let us look at some symbols commonly associated with mathematical operations.

The symbol $+$.

Words associated with this symbol are 'plus', 'add', 'increase' and 'positive'.

As it stands, '$+$' clearly has some sort of meaning, but we really need to understand it within a context.

So, for example, if we see the + symbol written in the sum

$$2 + 3$$

we understand that the context is one of adding the two numbers, 2 and 3, to give 5. So here, the symbol + is an instruction to add two numbers together.

Let us look at another context in which we see the $+$ symbol.

If you study telephone numbers on business cards you will often see them given, for example, as

$$+44\ 191\ 123\ 4567$$

In this context, the + symbol means that, in addition to the usual telephone number, a person dialing that number from overseas will need to include the country code (in this case 44).

So we see that the $+$ symbol can have completely different meanings in different contexts, and it is important to be clear about the context.

The symbol $-$.

Words associated with this symbol are 'minus', 'subtract', 'take away', 'negative' and 'decrease'.

Again, to understand the symbol we need a context.

So, if we see the – symbol written in the sum

$$6 - 4$$

we know this means 6 subtract 4, and we know the answer is 2.

In a different context, we might see -5°C, meaning a temperature of minus five degrees Celsius, that is five degrees below zero.

The symbol $\times$.

Words associated with this symbol are 'multiply', 'lots of', and 'times'.

This is really just a shorthand for adding. For example, if we see

$$6 + 6 + 6 + 6 + 6$$

we have five lots of six, or five sixes, and in our shorthand we can write this as 5×6.

Suppose we have

$$a + a + a + a + a$$

We might write this expression as $5 \times a$. However, in this context, especially in hand-written work, we may confuse the $\times$ symbol with the letter x, and so we would often write simply $5a$. We see that our shorthand has become even shorter. Multiplication is one of those rare occasions when we can omit a symbol altogether.

The division symbols

Division is symbolised in several different ways. For example

$$10 \div 5, \qquad \frac{10}{5}, \qquad 10/5$$

are three equivalent ways of writing ten divided by 5. We might also read this as 'how many times will 5 go into 10 ?'.

The = sign and its variants

Another symbol used frequently is the equals sign $=$.

The = sign does not mean anything on its own - we need a context.

For example, in the sum $1 + 2 = 3$, what we are saying is that whatever we have on the left-hand side is exactly equal to whatever we have on the right-hand side.

Variations on the equals sign are

$\neq$ which means 'is not equal to'

$\approx$ which means 'is approximately equal to'

$\geq$ which means 'is greater than or equal to',
e.g. $x \geq 2$ means that x can equal 2, but it might also be any value larger than 2.

$\leq$ which means 'is less than or equal to',
e.g. $y \leq 7$ means that y might equal 7 or might be any number less than 7.

3. Variables

Variables are another form of mathematical symbol. These are used when quantities take different values.

Imagine taking a car journey and think about the speed at which you are travelling. As you travel along your speed may change. So, speed is a variable - that is, a quantity which can change. We will be using letters to stand for quantities like this. For example, we might use the letter v for speed. To a large extent we can use any letter we choose, although there are conventions.

We might choose to use d for distance and t for time.

By convention, we use u to be an initial speed, and v to be a final speed.

In a different context, v might refer to volume. We need to check the context to fully understand the meaning.

If we see $v = \frac{d}{t}$ where d = distance, and t = time, then we would know that v is a speed.

On the other hand, if we see $v = \frac{4}{3}\pi r^3$ where r is the radius of a sphere, we know that v stands for the volume of the sphere.

Returning to our car journey. We might want to record the journey time on several different days. In this context we might choose to use a subscript and write

$$t_1,\ t_2,\ t_3,\ t_4,\ t_5$$

for the journey times on each of five different days. Alternatively we could write

$$t_m,\ t_t,\ t_w,\ t_h,\ t_f$$

for the journey times on Monday through to Friday. Note how we have used t_h for the journey time on Thursday to avoid confusion with t_t for the journey time on Tuesday.

So, a subscript is a small number, or other symbol, written to the bottom right of a variable to distinguish different instances of that variable.

4. The Greek alphabet

You will find that Greek letters are used in many calculations.

For example, the Greek letter 'pi', written π, is used to represent the number 3.14159……. This number continues forever without repeating.

We often use α ('alpha'), β ('beta'), and θ ('theta') to represent angles.

The Greek capital letter 'sigma' or Σ is frequently used to represent the addition of several numbers, and you will see it provided for this purpose on the toolbar of any spreadsheet program.

For future reference the full alphabet is given here:

The Greek alphabet

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϱ	epsilon	N	ν	nu	Φ	φ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

5. Some more symbols

The positioning of numbers and symbols in relation to each other also gives meaning. For example, you will come across use of superscripts. These are small numbers or symbols written at the top right of another, as in 4^2. In this context 4^2 is a shorthand for 'four squared' or 4×4. Similarly 4^3 is shorthand for 'four cubed' or $4 \times 4 \times 4$.

On the other hand 32° can mean different things in different contexts. It might mean an angle of 32 degrees. It could mean 32 to the power zero, which is actually 1. In printed work characters with slightly different shapes and sizes are used - compare 32° meaning 32 degrees and 32^0 meaning 32 to the power 0. Clearly these have very different meanings! You need to know the context! 32°C is a temperature of 32 degrees Celsius.

What about 6,3 ? This could mean several things. But with brackets around, $(6, 3)$ can mean a pair of coordinates used to plot a point on a graph.

Brackets can mean a variety of different things. For example, in the study of probability you will come across expressions like $p(H) = \frac{1}{2}$- this means the probability of scoring a Head, when tossing a coin, is $\frac{1}{2}$. Again, knowing the context is vital.

The symbol % is a percentage sign and means 'out of 100' as in 90% meaning 'ninety out of one hundred'.

$\sqrt{}$ is a square root sign. For example $\sqrt{16}$ is the number which when multiplied by itself is 16, that is 4 or -4.

$\bar{x}$, read as 'x bar' is the mean of a set of numbers.

$1.\dot{3}$ is a recurring decimal sign meaning that the 3's continue forever, as in 1.33333...

$1.\dot{3}1\dot{7}$ means that the 317 goes on forever, as in 1.317317317...

6. Summary

In summary, mathematical symbols are a precise form of shorthand. They have to have meaning for you. To help with understanding you have *context* and *convention*.

Units and Prefixes

In the study of mechanics we come across many different quantities. Each quantity has its own units. We need to be able to work with these units.

The International System of Units

There are a number of different systems of units in use. However, only the modern metric system, known as the International System of Units (abbreviated SI from the French, Système International d'Unités) will be used in these leaflets. In nearly every mechanics problem we encounter the quantities mass, length and time. As shown in Table 1, the units of these quantities are defined to be kilograms, metres and seconds respectively; these are arbitrarily defined but have become accepted standard units. The units of many other quantities can be derived from physical laws. To illustrate this point consider the units of force in Table 1. The units of force are derived from Newton's second law (see mechanics sheet 2.2) which relates the quantity force (**F**) to mass (m) and acceleration (**a**) and can, for a body of constant mass, be expressed as **F** = m**a**. From this law we can determine what the units of force must be; acceleration is measured in units m s^{-2} and mass is measured in kg so force is measured in kg m s^{-2} which are called newtons (N) in mechanics.

Quantity	Dimensional Symbol	Unit	Symbol
Mass	M	kilogram	kg
Length	L	metre	m
Time	T	second	s
Force	F	newton	N (= m kg s^{-2})

Table 1: Fundamental quantities in mechanics

Prefixes

When a numerical unit is either very small or very large, the units used to define its size may be modified by using a prefix. A few of the prefixes used in the SI system of units are shown in Table 2. Each prefix represents a unit that, in most cases, moves the decimal point of a numerical quantity to every third place. There are 4 exceptions: the multiples deca and hecto and the submultiples centi and deci. In engineering and science the use of these prefixes is generally avoided, with the exception of some volume and area measurements.

Rules for the proper use of units and prefixes

1. A symbol is never written with a plural, s, as it may be confused with the unit for second, s.
2. Symbols are usually written in lower case unless they are named after individuals (for example, newton, N) or have a prefix larger than kilo, k (see Table 2).

3. To avoid confusion with prefix symbols, quantities that are defined by several units that are multiples of one another are separated by a space. For example, compare m s (metre-second) and ms (milli-second) - there is no space between the prefix and the unit but there is between the two separate types of unit.

4. The exponent of a unit that has a prefix refers to both the unit and its prefix. For example, $\mu N^2 = (\mu N)^2 = \mu N \times \mu N$

5. Physical constants and numbers that have several digits on either side of the decimal point should be written with a space (rather than a comma) between every three digits. For example, 65 823.315 856.

6. Numbers used in calculations should be represented in terms of their base or derived units by converting all prefixes to powers of 10, with the final result expressed as a single prefix. For example $3145000000m$ should be written $3.145Gm$.

7. Compound prefixes should be avoided. For example, kμs (kilo-micro-second) should be expressed as ms (milli-second) since 1kμs = $1(10^3)(10^{-6})$s = $1(10^{-3})$s = 1ms.

8. The minute, hour and second are retained despite not being in decimal form, i.e. 1 minute = 60 seconds, not 10 seconds or 100 seconds.

9. Plane angular measurement is made in radians (rad). However, in many engineering mechanics problems degrees are often used, where $180^\circ = \pi$ rad.

Prefix	SI Symbol	Multiplier	Exponential
tera	T	1 000 000 000 000	10^{12}
giga	G	1 000 000 000	10^9
mega	M	1 000 000	10^6
kilo	k	1 000	10^3
hecto	h	100	10^2
deca	da	10	10^1
deci	d	0.1	10^{-1}
centi	c	0.01	10^{-2}
milli	m	0.001	10^{-3}
micro	μ	0.000 001	10^{-6}
nano	n	0.000 000 001	10^{-9}
pico	p	0.000 000 000 001	10^{-12}

Table 2: Some prefixes used in mechanics.

Exercises

1. Write the following quantities in units with the appropriate prefixes:
a) 3142590 m b) 0.0000012 N c) 0.001 kg d) 987600 m e) 10022 N
2. Write the following quantities in units with only one prefix:
a) 2 Mpm b) 1.56 GkN c) 1.4 Tμs d) 1.08 GcN e) 18.56 cdam

Solutions

1. a) 3.14259 Mm b) 1.2 μN c) $1g$ d) 0.9876 Mm e) 10.022 kN
2. a) 2 μm b) 1.56 TN c) 1.4 Ms d) 10.8 MN e) 1.856 m

Rules of arithmetic

Evaluating expressions involving numbers is one of the basic tasks in arithmetic. But if an expression is complicated then it may not be clear which part of it should be evaluated first, and so some rules must be established. There are also rules for calculating with negative numbers.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- evaluate an arithmetic expression using the correct order of precedence;
- add and subtract expressions involving both positive and negative numbers;
- multiply and divide expressions involving both positive and negative numbers.

Contents

1. Introduction

In this unit we are going to recall the precedence rules of arithmetic which allow us to work out calculations which involve brackets, powers, $+$, $-$, $\times$ and $\div$ and let us all arrive at the same answer. Then we will go on to calculations involving positive and negative numbers, and generate and use the rules for adding, subtracting, multiplying and dividing them.

2. Order of precedence

Suppose we have this expression:

$$2 + 4 \times 3 - 1\,.$$

To work this out we can work from left to right:

First add, then multiply, and finally subtract:	$(+\ \times\ -)$	to get	17.
Or we could:	$(+\ -\ \times)$	to get	12.
Or we could:	$(\times\ +\ -)$	to get	13.

And so on. As you can see, you get different answers according to the order in which the operations are carried out. To prevent this from happening, there is an established order of precedence in which the operations must be done. The order that most people follow is this:

Anything in brackets must be done first. Then we evaluate any powers. Next we do any divisions and multiplications, working from left to right. And finally we do the additions and subtractions, again working from left to right.

This is hard to remember so here's an acronym to help you — BODMAS. It means

Brackets
pOwers
Division
Multiplication
Addition
Subtraction

where division and multiplication have the same priority, and also addition and subtraction have the same priority, so in each case we have bracketed them together. You should remember BODMAS, and this will give you the precedence rules to work out calculations involving brackets, powers, $\div$, $\times$, $+$ and $-$. So if we go back to our original expression $2 + 4 \times 3 - 1$, using BODMAS we can evaluate the expression and get a standard answer.

$$\begin{aligned} 2 + 4 \times 3 - 1 &= 2 + 12 - 1 \qquad (\times \text{ first}) \\ &= 14 - 1 \qquad (+ \text{ and } - \text{ next, so we do } + \text{ first}) \\ &= 13\,. \end{aligned}$$

Example

$$\begin{aligned} 2 \times (3 + 5) &= 2 \times 8 \qquad \text{(brackets first, then multiply)} \\ &= 16. \end{aligned}$$

Example

$$\begin{aligned} 9 - 6 + 1 &= 3 + 1 \quad \text{(left to right, as + and − have the same priority)} \\ &= 4. \end{aligned}$$

Example

$$\begin{aligned} 3 + 2^2 &= 3 + 4 \quad \text{(power then add)} \\ &= 7. \end{aligned}$$

Example

$$\begin{aligned} (3 + 2)^2 &= 5^2 \quad \text{(brackets, then power)} \\ &= 25. \end{aligned}$$

Notice that the final two examples are very similar, but having the brackets in the last one made a big difference to the answer. The square in the earlier one applies only to the 2, whereas the square in the later one applies to the $(3 + 2)$ because of the brackets.

Key Point

BODMAS is an acronyn that serves as a reminder of the order in which operations have to be carried out when working with equations and formulas:

Brackets pOwers Division Multiplication Addition Subtraction

where division and multiplication have the same priority, and so do addition and subtraction. If you have several operations of the same priority then you work from left to right.

Exercises

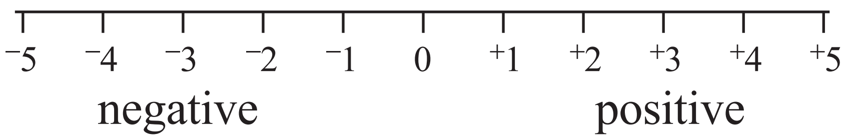
1. Find the value of the following expressions:

(a)	$2 \times 5 + 4$	(b)	$2 \times (5 + 4)$	(c)	$24 - 6 \div 2$	(d)	$3 + 4 \times (7 + 1)$
(e)	$(3 + 4) \times 7 + 1$	(f)	$5 + 2^2 \times 3$	(g)	$5 \times 2 - 4 \div 2$	(h)	$(3 + 2)^2$
(i)	$(5 + 4)^2 \times 4 \div 2$	(j)	$4 \times 2^2 - 12 \div 4$				

3. Adding and subtracting with negative numbers

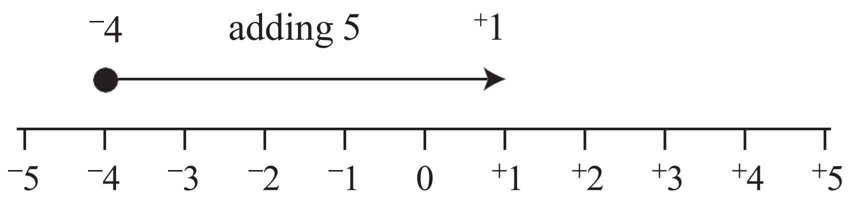
We now move on to other rules we use when working out calculations. What happens if we have calculations which involve positive and negative numbers? Are there any rules which help us?

We start with some revision — all real numbers are either positive or negative (or, of course, zero). The positive numbers are those greater than zero, and the negative ones are those less than zero. We can easily see this if we draw a number line and position zero in the middle. The numbers to the right are the *positive* numbers and the numbers to the left are *negative*.

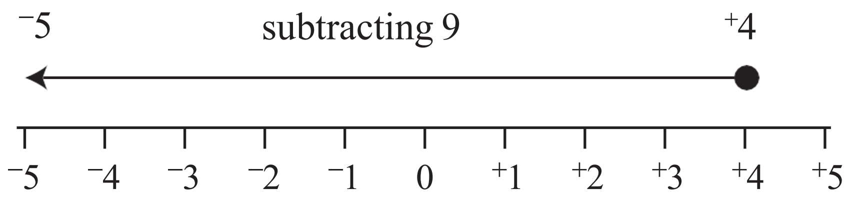


We have written positive three as ${}^{+}3$, and negative four as ${}^{-}4$. We use superscripts $^{+}$ and $^{-}$ so that they are not confused with the operations add (+) and subtract (−). So the superscripts help our understanding of what is going on, but with practice the standard notation is used and the superscripts are no longer needed. Also, as positive numbers are the most frequently used numbers it is not always necessary to include the positive sign — it can be omitted. So ${}^{+}3$ can be written as 3, and we know that it is 'positive three'.

So how do we add and subtract positive and negative numbers? Let's take some examples, using a number line. What is ${}^{-}4 + {}^{+}5$?



If we start at ${}^{-}4$ and count on five steps, you see that we get to ${}^{+}1$. And we use a similar idea for subtraction. What is ${}^{+}4 - {}^{+}9$?



Here, we start at ${}^{+}4$ and count back by nine steps, reaching ${}^{-}5$.

In these two examples we added and subtracted positive numbers. What happens if we add or subtract negative numbers? We can see by looking at the pattern in these additions.

$$
\begin{aligned}
5 + 2 &= 7\\
5 + 1 &= 6\\
5 + 0 &= 5\\
5 + {}^{-}1 &= 4\\
5 + {}^{-}2 &= 3\\
5 + {}^{-}3 &= 2\\
5 + {}^{-}4 &= 1 \qquad \text{etc.}
\end{aligned}
$$

Notice that the answers decrease by one, and the numbers added decrease by one each time. Also, look at the four last additions. Here we are adding negative numbers, but we can write these calculations as subtractions. They are subtractions of positive numbers, and give us the same answers.

$$5 + {}^{-}1 = 4 \text{ is the same as } 5 - 1 = 4$$
$$5 + {}^{-}2 = 3 \text{ is the same as } 5 - 2 = 3$$
$$5 + {}^{-}3 = 2 \text{ is the same as } 5 - 3 = 2$$
$$5 + {}^{-}4 = 1 \text{ is the same as } 5 - 4 = 1$$

So if we take two examples, $8 + {}^{-}10$ and ${}^{-}9 + {}^{-}5$, we can write them as subtractions of positive numbers and then calculate the answers by counting back:

$$8 + {}^{-}10 = 8 - 10 = {}^{-}2,$$
$${}^{-}9 + {}^{-}5 = {}^{-}9 - 5 = {}^{-}14$$

What about subtraction of negative numbers? Again we can use the number patterns to help us.

$$\begin{aligned} 4 - 2 &= 2 \\ 4 - 1 &= 3 \\ 4 - 0 &= 4 \\ 4 - {}^{-}1 &= 5 \\ 4 - {}^{-}2 &= 6 \\ 4 - {}^{-}3 &= 7 \\ 4 - {}^{-}4 &= 8 \end{aligned}$$

So here we have similar number patterns as before. As we subtract one less each time, the answers increase by one. But look at the last four subtractions. These are the same as additions of positive numbers.

$$4 - {}^{-}1 = 5 \text{ is the same as } 4 + 1 = 5$$
$$4 - {}^{-}2 = 6 \text{ is the same as } 4 + 2 = 6$$
$$4 - {}^{-}3 = 7 \text{ is the same as } 4 + 3 = 7$$
$$4 - {}^{-}4 = 8 \text{ is the same as } 4 + 4 = 8$$

So if we take two examples, $8 - {}^{-}10$ and ${}^{-}6 - {}^{-}12$, we can write them as additions of positive numbers and then calculate the answers by counting on:

$$8 - {}^{-}10 = 8 + 10 = 18,$$
$${}^{-}6 - {}^{-}13 = {}^{-}6 + 13 = 7.$$

So to add and subtract positive and negative numbers, here is the rule to remember. If the operation and the sign are the same, they work like the addition of a positive number. So

$+^{+}$ and $-^{-}$ work like the addition of a positive number.

If the operation and the sign are different, they work like the subtraction of a positive number. So

$+^{-}$ and $-^{+}$ work like the subtraction of a positive number.

Key Point

All real numbers are either positive or negative. Number lines can be used to show the positions of positive and negative numbers.

Positive three may be written as ${}^{+}3$ and negative three may be written as ${}^{-}3$ to distinguish the signs from the operations add (+) and subtract (–). Note that positive numbers are the most commonly used, and so the positive sign is usually omitted, so that 3 is usually written instead of ${}^{+}3$.

When adding and subtracting positive and negative numbers it is useful to remember the following rules. If the operation and the sign are the same, they work like adding a (positive) number, so that

$$- \,{}^{-} \text{ works like } + \,{}^{+}.$$

If the operation and the sign are different, they work like subtracting a (positive) number, so that

$$+ \,{}^{-} \text{ works like } - \,{}^{+}.$$

Exercises

2. Complete the following:

(a)	${}^{-}5 + {}^{+}8 =$	(b)	${}^{-}4 - 3 =$	(c)	$6 + {}^{-}3 =$	(d)	${}^{-}6 - {}^{-}3 =$
(e)	$13 + {}^{-}3 =$	(f)	${}^{-}20 - {}^{-}42 =$	(g)	$8 - 16 =$	(h)	$10 - {}^{-}23 =$
(i)	${}^{-}4 + {}^{-}2 - 5 =$	(j)	$9 - {}^{-}4 + {}^{-}5 =$				

4. Multiplying and dividing with negative numbers

Now that we have used addition and subtraction with both positive and negative numbers, what happens when we multiply or divide them? Are there any rules to help us?

We know how to multiply and divide pairs of positive numbers, for example

$$5 \times 5 = 25$$

and

$$5 \div 5 = 1.$$

So when we multiply and divide pairs of positive numbers, the answer is always a positive number. But what happens when we multiply and divide using negative numbers? What are the rules?

Let us look at some patterns again.

$$\begin{aligned} 5 \times 4 &= 20 \\ 5 \times 3 &= 15 \\ 5 \times 2 &= 10 \\ 5 \times 1 &= 5 \\ 5 \times 0 &= 0 \end{aligned}$$

So, using the pattern,

$$\begin{aligned} 5 \times {}^{-}1 &= {}^{-}5 \\ 5 \times {}^{-}2 &= {}^{-}10 \\ 5 \times {}^{-}3 &= {}^{-}15 \\ 5 \times {}^{-}4 &= {}^{-}20 \end{aligned}$$

Therefore when we multiply a positive number by a negative number, the answer is negative. As it doesn't matter which way round you multiply the numbers, we can also say that if we multiply a negative number by a positive number then the answer is negative. In short, when multiplying, if the two signs are different then the answer is negative.

$$(+) \times (-) = (-), \qquad (-) \times (+) = (-).$$

So we can now work out the following examples:

$$\begin{aligned} 6 \times {}^{-}5 &= {}^{-}30, \\ {}^{-}4 \times 3 &= {}^{-}12. \end{aligned}$$

But what if the two numbers are negative? Look at these multiplications and see if you can work out the rule. We know that

$$\begin{aligned} {}^{-}5 \times 4 &= {}^{-}20 \\ {}^{-}5 \times 3 &= {}^{-}15 \\ {}^{-}5 \times 2 &= {}^{-}10 \\ {}^{-}5 \times 1 &= {}^{-}5 \\ {}^{-}5 \times 0 &= 0 \end{aligned}$$

so, using the pattern,

$$\begin{aligned} {}^{-}5 \times {}^{-}1 &= 5 \\ {}^{-}5 \times {}^{-}2 &= 10 \\ {}^{-}5 \times {}^{-}3 &= 15 \\ {}^{-}5 \times {}^{-}4 &= 20 \end{aligned}$$

So we can see from the pattern that, when multiplying two negative numbers together, the answer is always positive. For example,

$$\begin{aligned} {}^{-}6 \times {}^{-}3 &= 18, \\ {}^{-}9 \times {}^{-}2 &= 18. \end{aligned}$$

But also remember that we get a positive answer when we multiply two positive numbers together:

$$\begin{aligned} 6 \times 3 &= 18, \\ 9 \times 2 &= 18 \end{aligned}$$

as well. So when we multiply if the signs are the same the answer is positive.

As division is the inverse of multiplication, the rules for division are the same as the rules for multiplication. So when multiplying and dividing positive and negative numbers remember this:

If the signs are the same the answer is positive,
if the signs are different the answer is negative.

(+) multiply or divide (+) (−) multiply or divide (−)	answer is (+),
(+) multiply or divide (−) (−) multiply or divide (+)	answer is (−).

So, for example,

$$\begin{aligned} {}^{-}6 \div {}^{-}2 &= 3 & &\text{(signs the same so positive),} \\ {}^{-}12 \div 3 &= {}^{-}4 & &\text{(signs different, so negative).} \end{aligned}$$

Key Point

When multiplying pairs of positive and negative numbers it is helpful to remember the following rules:

When the signs of the numbers are the same the answer is a positive number,

$$\left.\begin{array}{c} (+) \times (+) \\ (-) \times (-) \\ (+) \div (+) \\ (-) \div (-) \end{array}\right\} \text{answer is } (+).$$

When the signs of the numbers are different the answer is a negative number,

$$\left.\begin{array}{c} (+) \times (-) \\ (-) \times (+) \\ (+) \div (-) \\ (-) \div (+) \end{array}\right\} \text{answer is } (-).$$

Exercises

3. Complete the following:

(a)	$^{-}5 \times {}^{-}8 =$	(b)	$^{+}5 \times {}^{-}6 =$	(c)	$^{-}6 \times {}^{+}7 =$	(d)	$^{-}9 \times {}^{-}3 =$
(e)	$8 \times 9 =$	(f)	$^{+}16 \div {}^{+}2 =$	(g)	$^{-}24 \div 3 =$	(h)	$42 \div {}^{-}6 =$
(i)	$^{-}36 \div {}^{-}4 =$	(j)	$^{-}12 \div 3 \times {}^{-}2 =$				

Answers

1.

(a)	14	(b)	18	(c)	21	(d)	35	(e)	50
(f)	17	(g)	8	(h)	25	(i)	162	(j)	13

2.

(a)	3 or $^{+}3$	(b)	$^{-}7$	(c)	3 or $^{+}3$	(d)	$^{-}3$	(e)	10 or $^{+}10$
(f)	22 or $^{+}22$	(g)	$^{-}8$	(h)	33 or $^{+}33$	(i)	$^{-}11$	(j)	8 or $^{+}8$

3.

(a)	40 or $^{+}40$	(b)	$^{-}30$	(c)	$^{-}42$	(d)	27 or $^{+}27$	(e)	72 or $^{+}72$
(f)	8 or $^{+}8$	(g)	$^{-}8$	(h)	$^{-}7$	(i)	9 or $^{+}9$	(j)	8 or $^{+}8$

Polynomial division

mc-TY-polydiv-2009-1

In order to simplify certain sorts of algebraic fraction we need a process known as polynomial division. This unit describes this process.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that all this becomes second nature. To help you to achieve this, the unit includes a number of such exercises.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- simplify algebraic fractions by performing polynomial division

Contents

1. Introduction

In order to simplify certain sorts of algebraic fraction we need a process known as polynomial division. This unit describes how this process is carried out.

2. Long Division of numbers

The process is very like the long division of numbers. Let us look in detail at a long division sum and try to see how the process works. You should work through the example yourself since we are going to examine exactly how it was done.

Example

Suppose we wish to divide 2675 by 25. The calculation is set out like this:

$$\begin{array}{r} 107 \\ 25 \overline{)\,2675} \\ \underline{25} \\ 175 \\ \underline{175} \\ 0 \end{array}$$

Now how was it done ?

The first question was 'how many times does 25 divide into 26 ?'. The answer is 1 and this is recorded above the 6. We then multiplied the 25 by 1 and wrote the answer down beneath the 26. This is so that by subtracting the answer from 26 we can find out how much is left over after the division of 26 by 25. The amount left over, or remainder, is 1. Given its position the one remainder is not 1 unit but 1 hundred.

We now bring down the next figure in 2675, the 7, and set it alongside the 1 to give 17. The question we ask now is 'how many times does 25 divide into 17 ?'. The answer is none, which we record with 0 above the 7.

We now bring down the next figure in 2675, the 5, and set it alongside the 17 to give 175. The question we ask now is 'how many times does 25 divide into 175 ?'. We guess the answer is 7, which we record above the 5 of 2675.

This guess is important for what follows in polynomial division. We could base the guess on a rough estimate, how many times does 2 divide into 17, and the answer is 8. However 8 multiplied by 25 is 200, way over 175. Thus we go with 7.

We now multiply 25 by 7 to get 175 and write this below the 175 already there. We are seeking to find out how much is left over, or the remainder after the division has been completed, and so we subtract 175 from 175 giving an answer of 0, i.e. there is no remainder and the division is finished.

We see that 25 divides into 2675 exactly, 107 times, with no remainder, that is $\frac{2675}{25} = 107$.

3. Polynomial division

We now do the same process with algebra.

Example

Suppose we wish to find

$$\frac{27x^3 + 9x^2 - 3x - 10}{3x - 2}$$

The calculation is set out as we did before for long division of numbers:

$$3x - 2 \overline{\big)\, 27x^3 + 9x^2 - 3x - 10}$$

The question we ask is 'how many times does $3x$, NOT $3x - 2$, go into $27x^3$?'. The answer is $9x^2$ times. And we record this above the x^2 place, just as we did with the numbers:

$$\begin{array}{r} 9x^2 \qquad\qquad\quad \\ 3x - 2 \overline{\big)\, 27x^3 + 9x^2 - 3x - 10} \end{array}$$

Just as we did with the numbers we need to find the remainder, and so we multiply $9x^2$ by $3x-2$ and write the answer down under $27x^3 + 9x^2$. Thus we get:

$$\begin{array}{r} 9x^2 \qquad\qquad\quad \\ 3x - 2 \overline{\big)\, 27x^3 + 9x^2 - 3x - 10} \\ \underline{27x^3 - 18x^2} \qquad\qquad\quad \end{array}$$

To find out what is left we now subtract, and get $27x^2$, and as with the numbers we now bring down the next term, the $-3x$, and write it alongside the $27x^2$ to give:

$$\begin{array}{r} 9x^2 \qquad\qquad\quad \\ 3x - 2 \overline{\big)\, 27x^3 + 9x^2 - 3x - 10} \\ \underline{27x^3 - 18x^2} \qquad\qquad\quad \\ 27x^2 - 3x \qquad\quad \end{array}$$

The question to be asked now is, 'how many times does $3x$, NOT $3x - 2$, go into $27x^2$?'. The answer is $9x$ and we write this above the $-3x$ term, i.e. above the x place:

$$\begin{array}{r} 9x^2 + 9x \qquad\quad \\ 3x - 2 \overline{\big)\, 27x^3 + 9x^2 - 3x - 10} \\ \underline{27x^3 - 18x^2} \qquad\qquad\quad \\ 27x^2 - 3x \qquad\quad \end{array}$$

Again we want to know what is left over from the division, so we multiply $3x - 2$ by the $9x$ and write the answer down so we can subtract it from $27x^2 - 3x$, giving us $15x$:

$$\begin{array}{r} 9x^2 + 9x \qquad\quad \\ 3x - 2 \overline{\big)\, 27x^3 + 9x^2 - 3x - 10} \\ \underline{27x^3 - 18x^2} \qquad\qquad\quad \\ 27x^2 - 3x \qquad\quad \\ \underline{27x^2 - 18x} \qquad\quad \\ 15x \qquad\quad \end{array}$$

The process is now repeated for the third time; bring down the next term, -10, write it next to the $15x$, and ask 'how many times does $3x$, not $3x - 2$, go into $15x$?'.

$$
\begin{array}{rl}
 & \,9x^2 + 9x \\
3x-2 & \overline{)\,27x^3 + 9x^2 - 3x - 10} \\
 & \;\underline{27x^3 - 18x^2} \\
 & \qquad\quad 27x^2 - 3x \\
 & \qquad\quad \underline{27x^2 - 18x} \\
 & \qquad\qquad\qquad 15x - 10
\end{array}
$$

$3x$ divides into $15x$ five times, and so we record this above the number place, above the 10. We multiply $3x - 2$ by 5 and write the answer down so we can subtract it from $15x - 10$ and see what the remainder is:

$$
\begin{array}{rl}
 & \,9x^2 + 9x + 5 \\
3x-2 & \overline{)\,27x^3 + 9x^2 - 3x - 10} \\
 & \;\underline{27x^3 - 18x^2} \\
 & \qquad\quad 27x^2 - 3x \\
 & \qquad\quad \underline{27x^2 - 18x} \\
 & \qquad\qquad\qquad 15x - 10 \\
 & \qquad\qquad\qquad \underline{15x - 10} \\
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

And in this case there is no remainder. Thus

$$\frac{27x^3 + 9x^2 - 3x - 10}{3x - 2} = 9x^2 + 9x + 5$$

Check back through the calculation again and compare it with 2675 divided by 25. You should see that effectively they are the same process.

Example

Suppose we wish to find

$$\frac{x^4 + x^3 + 7x^2 - 6x + 8}{x^2 + 2x + 8}$$

The calculation is set out as follows, and an explanation is given below.

$$
\begin{array}{rl}
 & \;x^2 - x + 1 \\
x^2 + 2x + 8 & \overline{)\,x^4 + x^3 + 7x^2 - 6x + 8} \\
 & \;\underline{x^4 + 2x^3 + 8x^2} \\
 & \qquad -x^3 - x^2 - 6x \\
 & \qquad \underline{-x^3 - 2x^2 - 8x} \\
 & \qquad\qquad\quad x^2 + 2x + 8 \\
 & \qquad\qquad\quad \underline{x^2 + 2x + 8} \\
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Work through the example. Your thinking should be moving along the lines:

'How many times does x^2 go into x^4 ?'. The answer is x^2 times.

Write x^2 above the x^2 place in $x^4 + x^3 + 7x^2 - 6x + 8$.

Multiply $x^2 + 2x + 8$ by x^2, write the answer down underneath $x^4 + x^3 + 7x^2$ and subtract to find the remainder - which is $-x^3 - x^2$.

Bring down the next term, $-6x$, to give $-x^3 - x^2 - 6x$.

'How many times does x^2 go into $-x^3$?'. The answer is $-x$ times.

Write $-x$ above the x place in $x^4 + x^3 + 7x^2 - 6x + 8$

Multiply $x^2 + 2x + 8$ by $-x$, write the answer underneath $-x^3 - x^2 - 6x$ and subtract to find the remainder, which is $x^2 + 2x$.

Bring down the next term, 8, to give $x^2 + 2x + 8$.

'How many times does x^2 go into x^2 ?' The answer is 1.

Write 1 above the number place in $x^4 + x^3 + 7x^2 - 6x + 8$.

Multiply $x^2 + 2x + 8$ by 1, write the answer down underneath $x^2 + 2x + 8$ and subtract to find the remainder, which is 0.

So we conclude that

$$\frac{x^4 + x^3 + 7x^2 - 6x + 8}{x^2 + 2x + 8} = x^2 - x + 1$$

Example

What happens if some of the terms are missing from the polynomial into which we are dividing? The answer is that we leave space for them when we set out the division and write in the answers to the various small divisions that we do in the places where they would normally go.

Suppose we wish to find

$$\frac{x^3 - 1}{x - 1}$$

The calculation is set out as follows:

$$\begin{array}{r|llll}
 & x^2 & +\ x & +\ 1 & \\
\hline
x - 1 & x^3 & & & -\ 1 \\
 & x^3 - x^2 & & & \\
\cline{2-2}
 & \quad x^2 & & & \\
 & \quad x^2 - x & & & \\
\cline{2-3}
 & & x - 1 & & \\
 & & x - 1 & & \\
\cline{3-4}
 & & & 0 &
\end{array}$$

We conclude that

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1$$

Example

What would we do if there was a final remainder, something other than 0 ?

For example, suppose we needed to find

$$\frac{27x^3 + 9x^2 - 3x - 9}{3x - 2}$$

We do exactly the same:

$$\begin{array}{r|llll}
 & & 9x^2 & +\ 9x & +\ 5 \\
\hline
3x - 2 & 27x^3 & +\ 9x^2 & -\ 3x & -\ 9 \\
 & 27x^3 & -\ 18x^2 & & \\
\cline{2-3}
 & & 27x^2 & -\ 3x & \\
 & & 27x^2 & -\ 18x & \\
\cline{3-4}
 & & & 15x & -\ 9 \\
 & & & 15x & -\ 10 \\
\cline{4-5}
 & & & & 1
\end{array}$$

We now have a remainder of 1, which still has to be divided by $3x-2$. Thus our final answer now is:

$$\frac{27x^3+9x^2-3x-9}{3x-2} = 9x^2+9x+5+\frac{1}{3x-2}$$

Exercises

Use polynomal division to simplify each of the following quotients.

a) $\dfrac{x^4+3x^3-x^2-x+6}{x+3}$ b) $\dfrac{2x^4-5x^3+2x^2+5x-10}{x-2}$ c) $\dfrac{7x^4-10x^3+3x^2+3x-3}{x-1}$

d) $\dfrac{2x^4+8x^3-5x^2-4x+2}{x^2+4x-2}$ e) $\dfrac{3x^4-x^3+8x^2+5x+3}{x^2-x+3}$ f) $\dfrac{3x^4+9x^3-5x^2-6x+2}{3x^2-2}$

g) $\dfrac{x^3-2x^2-4}{x-2}$ h) $\dfrac{x^3-4x^2+9}{x-3}$ i) $\dfrac{x^4-13x-42}{x^2-x-6}$

Answers

a) x^3-x+2 b) $2x^3-x^2+5$ c) $7x^3-3x^2+3$

d) $2x^2-1$ e) $3x^2+2x+1$ f) x^2+3x-1

g) x^2+2x+2 h) x^2-x-3 i) x^2+x+7

Fractions: adding and subtracting

In this unit we shall see how to add and subtract fractions. We shall also see how to add and subtract mixed fractions by turning them into improper fractions.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- add and subtract fractions;
- add and subtract mixed fractions.

Contents

1. Introduction

Here is a simple example of adding fractions: calculate $\frac{1}{5} + \frac{2}{5}$.

To understand this, suppose we have a cake and divide it into five equal pieces. Each piece is a fifth, or $\frac{1}{5}$, of the cake. If we take one fifth, and then a further two fifths, we have taken a total of three fifths:

$$\frac{1}{5} + \frac{2}{5} = \frac{3}{5} .$$

Example
Calculate $\frac{1}{8} + \frac{1}{8} + \frac{5}{8}$.

Solution

$$\frac{1}{8} + \frac{1}{8} + \frac{5}{8} = \frac{7}{8} .$$

In both of these examples we were adding 'like' things. In the first example we were adding fifths, and in the second we were adding eighths. So in both cases the denominators were the same. So to add 'like' fractions we just add the numerators.

The process is similar for subtraction, but we take away instead of adding.

Example
Calculate $\frac{5}{8} - \frac{3}{8}$.

Solution

$$\frac{5}{8} - \frac{3}{8} = \frac{2}{8} = \frac{1}{4} .$$

Example
Calculate $\frac{3}{5} + \frac{4}{5}$.

Solution

$$\frac{3}{5} + \frac{4}{5} = \frac{7}{5} = 1\tfrac{2}{5} .$$

Key Point

When adding or subtracting 'like' fractions, when the denominators are the same, just add or subtract the numerators.

Exercises

1. Add the following fractions:

(a) $\frac{1}{6}+\frac{3}{6}$ (b) $\frac{1}{8}+\frac{1}{8}+\frac{3}{8}+\frac{5}{8}$ (c) $\frac{3}{1}+\frac{4}{10}+\frac{7}{10}$

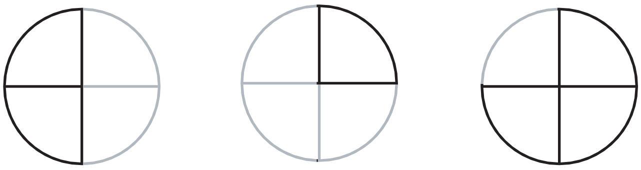
2. Subtract the following fractions:

(a) $\frac{7}{8}-\frac{3}{8}$ (b) $\frac{9}{1}-\frac{2}{1}$ (c) $\frac{11}{1}-\frac{4}{1}$

2. Fractions with different denominators

What happens when we want to add or subtract fractions where the denominators are not the same? Let us look at a simple case. What is $\frac{1}{2}+\frac{1}{4}$?

If we think of a pizza cut in half and then into quarters, we can see that if we take a half and then a quarter we will have taken a total of three quarters.

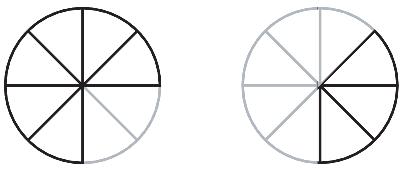


So $\frac{1}{2}$ is equivalent to $\frac{2}{4}$, and then adding $\frac{1}{4}$ gives us $\frac{3}{4}$ in total:

$$\frac{1}{2}+\frac{1}{4} = \frac{2}{4}+\frac{1}{4} = \frac{3}{4}.$$

Example
Calculate $\frac{3}{4}+\frac{3}{8}$.

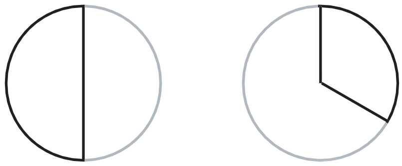
Solution



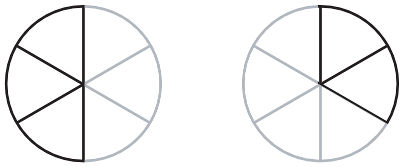
If we change the quarters into eighths, it becomes straightforward. The fraction $\frac{3}{4}$ is equivalent to the fraction $\frac{6}{8}$ and since the fractions now have the same denominator, we can just add the numerators:

$$\frac{3}{4}+\frac{3}{8} = \frac{6}{8}+\frac{3}{8} = \frac{9}{8} = 1\frac{1}{8}.$$

So far our examples have used fractions within the same family, where it is easy to see a connection between the fractions. For instance, quarters fit exactly into a half, and eighths fit exactly into a quarter. We shall now look at what happens when we add $\frac{1}{2}$ and $\frac{1}{3}$



This time $\frac{1}{2}$ will not fit exactly into $\frac{1}{3}$, nor will $\frac{1}{3}$ fit exactly into a $\frac{1}{2}$. So we need to find a number that can be divided exactly by both 2 and 3, and then split each whole into that number of pieces. Now 6 can be divided by both 2 and 3, so if we split each whole into 6 pieces then we can see that $\frac{1}{2}$ is $\frac{3}{6}$ and $\frac{1}{3}$ is $\frac{2}{6}$.



So we have

$$\frac{1}{2} + \frac{1}{3} \quad = \quad \frac{3}{6} + \frac{2}{6} \quad = \quad \frac{5}{6}.$$

Example

Calculate $\frac{1}{4} + \frac{2}{}$.

Solution

Again quarters and fifths are different sizes of fraction, and we cannot exactly fit quarters into fifths or fifths into quarters. So we need to find a size of fraction that will fit into both quarters and fifths.

Let us start by listing some numbers that can be divided by 4:

$$4, 8, 12, 16, 20, 24, \ldots.$$

And here are some numbers that can be divided by 5:

$$5, 10, 15, 20 —$$

and now we see that 20 can be divided by both 4 and 5. It is the smallest number in both the lists, so we shall split both wholes into 20 equal pieces.

If we look at this numerically, what we are doing is finding the smallest number that can be divided by the two denominators. The denominators are 4 and 5, so the number we take is 20. Then we convert the two fractions into equivalent fractions with the same denominator, 20, before adding them. We say that 20 is the *common denominator*. To find the first equivalent fraction we see how many times 4 goes into 20. It goes 5 times, so we multiply both the numerator, 1, and the denominator, 4, by 5. To find the second equivalent fraction, we see how many times 5 goes into 20. It goes 4 times, so we multiply both the numerator, 2, and the denominator, 5, by 4:

$$\frac{1}{4} + \frac{2}{5} \quad = \quad \frac{1 \times 5}{4 \times 5} + \frac{2 \times 4}{5 \times 4} \quad = \quad \frac{5}{20} + \frac{8}{20} \quad = \quad \frac{13}{20}.$$

Example

Calculate $\frac{3}{4} - \frac{1}{}$.

Solution

To carry out this calculation, we must find the smallest number that can be divided by both 4 and 6. That number is 12, so we need to convert both our fractions in to twelfths:

$$\frac{3}{4} - \frac{1}{6} \quad = \quad \frac{3 \times 3}{4 \times 3} - \frac{1 \times 2}{6 \times 2} \quad = \quad \frac{9}{12} - \frac{2}{12} \quad = \quad \frac{7}{12}.$$

In all these cases we have been changing the fractions into equivalent fractions before adding or subtracting. The denominator of the equivalent fraction is chosen so that it is the lowest number that can be divided by the other denominators, and it is called the *lowest common denominator*, or l.c.d. In some cases the l.c.d. can easily be found by multiplying together the denominators of the fractions to be added or subtracted. But, as our last example shows, doing this does not always result in the l.c.d. As you can see, if we had taken 4×6 and used 24 as our common denominator, the result would have been $\frac{14}{2}$ and we would then have needed to find the lowest form of the fraction by dividing both numerator and denominator by any common factors of 14 and 24.

3. Mixed fractions

Now let us look at how to add and subtract mixed fractions. Take, for example, $5\frac{3}{4} - 1\frac{4}{5}$.

To add or subtract mixed fractions, we turn them into improper fractions first. So

$$5\frac{3}{4} - 1\frac{4}{5} \quad = \quad \frac{5 \times 4 + 3}{4} - \frac{1 \times 5 + 4}{5}$$
$$= \quad \frac{23}{4} - \frac{9}{5}.$$

Now the improper fractions are treated just the same as before. We find the lowest common denominator of 4 and 5. The l.c.d. is 20, so

$$\frac{23}{4} - \frac{9}{5} \quad = \quad \frac{23 \times 5}{4 \times 5} - \frac{9 \times 4}{5 \times 4} \quad = \quad \frac{115}{20} - \frac{36}{20} \quad = \quad \frac{79}{20} \quad = \quad 3\frac{19}{20}.$$

Example

Calculate $1\frac{3}{4} + 6\frac{2}{5} + \frac{5}{2}$.

Solution

First of all, write all the mixed fractions as improper fractions:

$$1\frac{3}{4} + 6\frac{2}{5} + \frac{5}{2} = \frac{1 \times 4 + 3}{4} + \frac{6 \times 5 + 2}{5} + \frac{5}{2}$$
$$= \frac{7}{4} + \frac{32}{5} + \frac{5}{2}.$$

We now want the lowest common denominator of 2, 4 and 5. An easy way of finding this is to count up in multiples of the largest denominator, in this case 5, see whether the other denominators, 2 and 4, are factors. So 5 is no good, 10 is no good, 15 is no good, but 20 fits our requirements. So

$$\frac{7}{4} + \frac{32}{5} + \frac{5}{2} = \frac{7 \times 5}{4 \times 5} + \frac{32 \times 4}{5 \times 4} + \frac{5 \times 10}{2 \times 10} = \frac{35}{20} + \frac{128}{20} + \frac{50}{20} = \frac{213}{20}.$$

We can then turn the answer back into a mixed fraction by dividing by the denominator and finding the remainder: $213 \div 20$ equals 10 remainder 13, so the answer is $10\frac{13}{20}$.

Key Point

Turn mixed fractions to improper fractions before adding or subtracting them.

Exercises

3. Perform the following calculations:

(a) $\frac{1}{2} + \frac{1}{5}$ (b) $\frac{2}{3} + \frac{5}{9}$ (c) $\frac{2}{7} + \frac{3}{4}$ (d) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

(e) $2\frac{3}{5} - \frac{4}{3}$ (f) $3\frac{2}{3} - 1\frac{1}{4}$ (g) $1\frac{1}{2} - \frac{7}{10}$ (h) $4\frac{1}{4} - \frac{2}{5} - \frac{1}{8}$

Answers

1.

(a) $\frac{2}{3}$ (b) $\frac{5}{4}$ or $1\frac{1}{4}$ (c) $\frac{7}{5}$ or $1\frac{2}{5}$

2.

(a) $\frac{1}{2}$ (b) $\frac{7}{1}$ (c) $\frac{7}{1}$

3.

(a) $\frac{3}{1}$ (b) $\frac{11}{9}$ or $1\frac{2}{9}$ (c) $\frac{29}{2}$ or $1\frac{1}{2}$ (d) $\frac{13}{1}$ or $1\frac{1}{1}$

(e) $\frac{19}{1}$ or $1\frac{4}{1}$ (f) $\frac{29}{1}$ or $2\frac{5}{1}$ (g) $\frac{4}{5}$ (h) $\frac{149}{4}$ or $3\frac{29}{4}$

Fractions: multiplying and dividing

In this unit we shall see how to multiply fractions. We shall also see how to divide fractions by turning the second fraction upside down.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

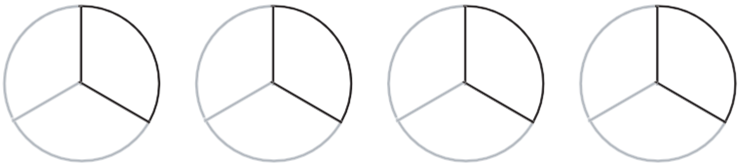
After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- multiply fractions, including improper fractions and mixed fractions;
- divide fractions, including improper fractions and mixed fractions.

Contents

1. Multiplying fractions

How do we multiply fractions? Let us start with an example. What is $4 \times \frac{1}{3}$? This means 4 lots of one third.



Numerically, we perform the calculation like this:

$$4 \times \frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{4}{3} = 1\tfrac{1}{3}\,.$$

The concept is the same as when we multiply whole numbers, for instance

$$4 \times 5 = 5 + 5 + 5 + 5 = 20\,.$$

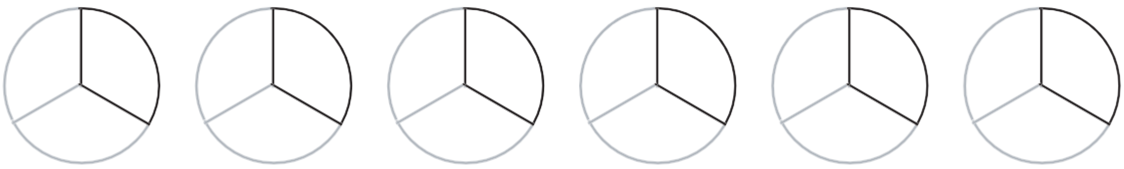
Now when we multiply numbers, we often use the fact that multiplication is 'commutative'. This means that, for example, $4 \times 5 = 5 \times 4$, so that

$$\begin{aligned} 5 + 5 + 5 + 5 &= 4 + 4 + 4 + 4 + 4\,, \\ 20 &= 20\,. \end{aligned}$$

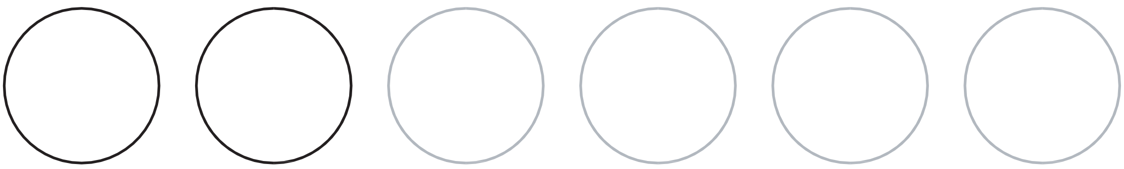
Let us see what this means when we are using fractions. We expect to find that $6 \times \frac{1}{3} = \frac{1}{3} \times 6$. Now

$$6 \times \frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\,,$$

and $(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}) + (\frac{1}{3} + \frac{1}{3} + \frac{1}{3})$ equals $1 + 1 = 2$.



On the other hand, $\frac{1}{3} \times 6$ means one-third of 6, and that is just 6 3, so that we have 6 wholes split into 3 equal parts, giving us 2 again.



So taking a fraction a whole number of times is the same as taking a fraction of a whole number.

Example

Calculate $5 \times \frac{2}{3}$.

Solution

We obtain

$$5 \times \frac{2}{3} = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{10}{3} = 3\tfrac{1}{3}.$$

This is just the same as two-thirds of 5. Now one-third of 5 is $\frac{5}{3}$, so two-thirds of 5 is $2 \times \frac{5}{3}$ which is $\frac{10}{3}$.

Any whole number can be written as a fraction. For example, 2 can be written as $\frac{2}{1}$, $\frac{4}{2}$, $\frac{6}{3}$, and so on. Any numbers can be used, as long as the numerator is twice the denominator. So, for example,

$$2 \times \frac{3}{4} = \frac{2}{1} \times \frac{3}{4} = \frac{2 \times 3}{1 \times 4} = \frac{6}{4} = \frac{3}{2} = 1\tfrac{1}{2}.$$

Example

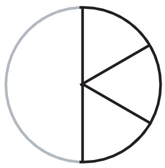
Calculate $7 \times \frac{5}{9}$.

Solution

We obtain

$$7 \times \frac{5}{9} = \frac{7}{1} \times \frac{5}{9} = \frac{7 \times 5}{1 \times 9} = \frac{35}{9} = 3\tfrac{8}{9}.$$

Now let us look at multiplying fractions by other fractions. For example, what is $\frac{1}{3} \times \frac{1}{2}$? This means one third of one half.

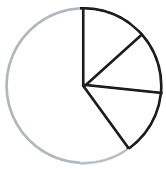


If we take a half and split it into 3, we have $\frac{1}{6}$ of a whole. We can see this if we also divide the other half into 3 pieces. Numerically, we have

$$\frac{1}{3} \times \frac{1}{2} = \frac{1 \times 1}{3 \times 2} = \frac{1}{6}.$$

As you can see, we have obtained the answer $\frac{1}{6}$ by first multiplying together the two numerators to give the numerator of the answer. We have then multiplied together the two denominators to give the denominator of the answer. This is the general technique we shall use.

Let us take another example. What is $\frac{1}{3} \times \frac{2}{5}$? This means that we want to start with two-fifths of a whole, and then take one third of that.



To split $\frac{2}{5}$ into 3 parts, it is easier to split each fifth into 3 parts and take one from each. Splitting each fifth into 3 gives us 15 pieces, so we have $\frac{1}{1}$ from each one fifth section, giving $\frac{2}{1}$:

$$\frac{1}{3} \times \frac{2}{5} = \frac{1 \times 2}{3 \times 5} = \frac{2}{15}.$$

So, as before, we are multiplying the numerators together and then multiplying the denominators together.

Example
Calculate $\frac{2}{5} \times \frac{4}{9}$.

Solution
We obtain

$$\frac{2}{5} \times \frac{4}{9} = \frac{2 \times 4}{5 \times 9} = \frac{8}{45}.$$

Example
Calculate $\frac{2}{3} \times \frac{4}{}$.

Solution
We obtain

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}.$$

Example
Calculate $\frac{2}{3} \times \frac{9}{10}$.

Solution
We obtain

$$\frac{2}{3} \times \frac{9}{10} = \frac{2 \times 9}{3 \times 10} = \frac{18}{30}.$$

Now in this example we can simplify the result by cancelling, and we would get $\frac{3}{5}$ as our final result. But often it is easier to cancel as we go along. If we do that, we get

$$\frac{2}{3} \times \frac{9}{10} = \frac{2 \times 9}{3 \times 10} = \frac{1 \times 3}{1 \times 5} = \frac{3}{5}.$$

The process is exactly the same if we wish to multiply three fractions rather than two.

Example
Calculate $\frac{1}{2} \times \frac{3}{4} \times \frac{2}{3}$.

Solution
We obtain

$$\frac{1}{2} \times \frac{3}{4} \times \frac{2}{3} = \frac{1 \times 3 \times 2}{2 \times 4 \times 3} = \frac{1 \times 1 \times 1}{1 \times 4 \times 1} = \frac{1}{4}.$$

What happens if we have mixed fractions?

Just as before when dealing with mixed fractions, we turn them into improper fractions first.

Example

Calculate $2\frac{1}{3} \times \frac{3}{4}$.

Solution

We obtain

$$2\frac{1}{3} \times \frac{3}{4} = \frac{2 \times 3 + 1}{3} \times \frac{3}{4} = \frac{7 \times 3}{3 \times 4} = \frac{7 \times 1}{1 \times 4} = \frac{7}{4} = 1\frac{3}{4}.$$

Example

Calculate $1\frac{2}{5} \times 2\frac{5}{6}$.

Solution

We obtain

$$1\frac{2}{5} \times 2\frac{5}{6} = \frac{1 \times 5 + 2}{5} \times \frac{2 \times 6 + 5}{6} = \frac{7 \times 17}{5 \times 6} = \frac{119}{30} = 3\frac{29}{3}.$$

Key Point

To multiply fractions, multiply the numerators together and multiply the denominators together separately. To multiply mixed fractions, turn them into improper fractions first.

Exercises

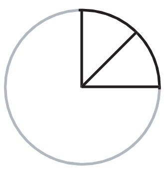
1. Perform the following multiplications:

(a) $6 \times \frac{2}{5}$ (b) $2 \times \frac{5}{9}$ (c) $4 \times \frac{3}{10}$ (d) $\frac{3}{4} \times \frac{7}{11}$

(e) $\frac{3}{5} \times \frac{10}{1}$ (f) $2\frac{1}{5} \times 3\frac{2}{3}$

2. Dividing fractions

We shall now look at what happens when we divide fractions. Let us take $\frac{1}{4}$ and divide it by 2.



$$\frac{1}{4} \div 2 = \frac{1}{8}$$

Dividing by 2 is the same as taking a half, so dividing by 2 and multiplying by a half are the same thing. We could write

$$\frac{1}{4} \div 2 = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}.$$

What about $\frac{1}{3}$ divided by 4? We would have

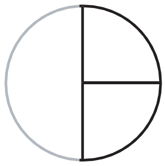
$$\frac{1}{3} \div 4 = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}.$$

And again, for $\frac{1}{2} \div 2$ we would have

$$\frac{1}{2} \div 2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

In all these cases it looks as if, to divide by a number, we can instead multiply the denominator by that number. Another way of saying this is that we write the divisor, that is the number we are dividing by, as a fraction. When we have done this, we turn that fraction upside-down and multiply instead.

Now we can extend this approach to divide by numbers which are themselves fractions rather than whole numbers. For example, what is $\frac{1}{2} \div \frac{1}{4}$? In other words, how many times does a quarter go into a half?



Following the rule, we obtain

$$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{1 \times 4}{2 \times 1} = \frac{4}{2} = 2$$

which makes sense since $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, so $\frac{1}{4}$ goes into $\frac{1}{2}$ twice.

Example
Calculate $\frac{1}{3} \div \frac{1}{5}$.

Solution

We obtain

$$\frac{1}{3} \div \frac{1}{5} = \frac{1}{3} \times \frac{5}{1} = \frac{1 \times 5}{3 \times 1} = \frac{5}{3} = 1\frac{2}{3}.$$

Example

Calculate $2 \div \frac{1}{8}$.

Solution

We obtain

$$2 \div \frac{1}{8} = 2 \times \frac{8}{1} = 2 \times 8 = 16.$$

Example

Calculate $4 \div \frac{1}{3}$.

Solution

We obtain

$$4 \div \frac{1}{3} = 4 \times \frac{3}{1} = \frac{12}{1} = 12.$$

So far, when dividing we have only looked at fractions with a numerator of 1. Let us now look at other fractions. What is $\frac{3}{4} \div 2$? Using the rule, we obtain

$$\frac{3}{4} \div 2 = \frac{3}{4} \div \frac{2}{1} = \frac{3}{4} \times \frac{1}{2} = \frac{3 \times 1}{4 \times 2} = \frac{3}{8}.$$

So again the process is turn the second fraction, the divisor, upside down, and then to multiply.

Example

Calculate $\frac{3}{5} \div 4$.

Solution

We obtain

$$\frac{3}{5} \div 4 = \frac{3}{5} \div \frac{4}{1} = \frac{3}{5} \times \frac{1}{4} = \frac{3 \times 1}{5 \times 4} = \frac{3}{20}.$$

Example

Calculate $\frac{2}{3} \div \frac{3}{4}$.

Solution

How many times does $\frac{3}{4}$ fit into $\frac{2}{3}$? Since $\frac{3}{4}$ is larger than $\frac{2}{3}$ we should expect the answer to be less than 1. Following the rule, we obtain

$$\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{2 \times 4}{3 \times 3} = \frac{8}{9}.$$

Finally we need to look at how to deal with mixed fractions when dividing.

Example

Calculate $1\frac{2}{3} \div 2\frac{1}{4}$.

Solution

As before, we need to convert mixed fractions to improper fractions before dividing:

$$1\tfrac{2}{3} \div 2\tfrac{1}{4} = \frac{1 \times 3 + 2}{3} \div \frac{2 \times 4 + 1}{4} = \frac{5}{3} \div \frac{9}{4}.$$

Now we continue as before by turning the divisor upside down, and multiplying:

$$\frac{5}{3} \div \frac{9}{4} = \frac{5}{3} \times \frac{4}{9} = \frac{20}{27}.$$

Example

Calculate $2\frac{4}{5} \div 4\frac{2}{3}$.

Solution

$$2\tfrac{4}{5} \div 4\tfrac{2}{3} = \frac{2 \times 5 + 4}{5} \div \frac{4 \times 3 + 2}{3} = \frac{14}{5} \div \frac{14}{3} = \frac{14}{5} \times \frac{3}{1} = \frac{3}{5}.$$

Key Point

To divide fractions, turn the second fraction upside-down and multiply. To divide mixed fractions, turn them into improper fractions first.

Exercises

2. Perform the following divisions:

(a) $\frac{1}{5} \div 2$ (b) $\frac{1}{6} \div 4$ (c) $\frac{1}{4} \div \frac{1}{2}$ (d) $\frac{1}{3} \div \frac{1}{3}$
(e) $2 \div \frac{1}{5}$ (f) $3 \div \frac{1}{4}$ (g) $\frac{2}{5} \div 3$ (h) $\frac{3}{4} \div 2$
(i) $\frac{3}{4} \div \frac{2}{3}$ (j) $\frac{3}{5} \div \frac{1}{2}$ (k) $4 \div \frac{2}{5}$ (l) $5 \div \frac{2}{3}$

3. Perform the following divisions:

(a) $2\frac{1}{10} \div 1\frac{1}{8}$ (b) $5\frac{1}{4} \div \frac{3}{8}$ (c) $1\frac{1}{3} \div 3\frac{1}{4}$

Answers

1.

(a) $\frac{12}{5}$ or $2\frac{2}{5}$ (b) $\frac{10}{9}$ or $1\frac{1}{9}$ (c) $\frac{6}{5}$ or $1\frac{1}{5}$ (d) $\frac{21}{4}$
(e) $\frac{1}{2}$ (f) $\frac{121}{1}$ or $8\frac{1}{1}$

2.

(a) $\frac{1}{1}$ (b) $\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 1
(e) 10 (f) 12 (g) $\frac{2}{1}$ (h) $\frac{3}{8}$
(i) $\frac{9}{8}$ or $1\frac{1}{8}$ (j) $\frac{6}{5}$ or $1\frac{1}{5}$ (k) 10 (l) $\frac{15}{2}$ or $7\frac{1}{2}$

3.

(a) $\frac{28}{1}$ or $1\frac{13}{1}$ (b) 14 (c) $\frac{16}{3}$

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Decimals

In this unit we shall look at the meaning of decimals, and how they are related to fractions. We shall then look at rounding to given numbers of decimal places or significant figures. Finally we shall take a brief look at irrational numbers.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- multiply and divide a decimal by powers of ten;
- convert decimals into fractions, and vice versa;
- round a decimal number to a given number of decimal places, or to a given number of significant figures.

Contents

1. Introduction

A simple meaning of the word 'decimal' is 'connected with ten', and the decimal number system is a means of expressing any number from the very smallest to the very largest.

Let us look at how the system works. We have the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, they are our units, or 'ones'. To make all our other numbers we use these same digits, but in different places in our decimal system. For example, in the number 12, '2' is in the units place and stands for 2 units, and '1' is in the tens place and stands for one times ten, giving us 2 + 10 = 12.

We can describe this by using a place value chart, and labelling the columns.

1000000	100000	10000	1000	100	10	1	• 1/10	1/100	1/1000

The decimal point shows us where the fraction part of a number begins. To the left of the decimal point we have the whole number part. We have the units, the tens, hundreds, thousands, ten thousands, hundred thousands, millions and so on. You can see that, as we go from right to left, each column heading is ten times larger than its neighbour, so going in this direction we are multiplying by ten each time. On the other hand, if we go from left to right, each column heading is ten times smaller than its neighbour, so going in this direction we are dividing by ten.

To the right of the decimal point we have the fractional part of the number. If we continue going right, we continue dividing by ten. So we get tenths, hundredths, thousandths and so on.

We can put some numbers in the chart.

1000000	100000	10000	1000	100	10	1	• 1/10	1/100	1/1000
					2	7			
				5	3	1			
					5	0			
			6	0	0	0			
				2	0	7			
	1	2	7	3	9	5			
						6	• 3	9	2
						0	• 5		
					1	2	• 0	2	7

These numbers are 27, 531, 50, 6 000, 207, 127 395, 6.392, 0.5, and 12.027. In the chart, the numbers are aligned so that the units are vertically above or below one another, and similarly for the tens, the hundreds, and so on. For instance, 50 has a 5 in the tens column. But we also put a 0 in the units column, to show that there are no units. This 0 is called a 'placeholder', because it makes sure we realise that the 5 represents 5 tens, and not 5 units.

2. Multiplying and dividing by ten

Using our chart, we can see what happens when we multiply a number by ten. We shall work out 34 × 10.

1000000	100000	10000	1000	100	10	1	•1/10	1/100	1/1000
					3	4			
				←	←				
				3	4	0			

The 3 representing the tens becomes ten times bigger, and moves to the hundreds column. Similarly, the 4 representing the units becomes ten times bigger and moves to the tens column. Now the units column is empty, and since there are no tenths in the original number to make ten times bigger, we need to put a zero in the units column.

Next, we shall take 0.507 and multiply it by 100.

1000000	100000	10000	1000	100	10	1	•1/10	1/100	1/1000
						0	•5	0	7
					←	←	←	←	
				←	←	←	←		
				0	5	0	•7		

Each digit will now move two columns to the left. In this case the leading zero is no longer required, so the result is 50.7 .

With division, the process is similar. We shall use the chart to work out 127.5 ÷ 10.

1000000	100000	10000	1000	100	10	1	•1/10	1/100	1/1000
				1	2	7	•5		
				→	→	→	→		
					1	2	•7	5	

The 1 in the hundreds column becomes ten times smaller, and moves to the tens column. The 2 in the tens column becomes ten times smaller, and moves to the units column. The 7 in the units column becomes ten times smaller, and moves to the tenths column. And finally, the 5 in the tenths column becomes ten times smaller, and moves to the hundredths column. So 127.5 ÷ 10 = 12.75 .

Next, we shall work out 2.3 ÷ 1000. Here, we need to use an extra column on the right of our chart to include ten-thousandths, and so to save space we shall omit the column for millions. Notice that 1000 is $10 \times 10 \times 10 = 10^3$.

100000	10000	1000	100	10	1	•1/10	1/100	1/1000	1/10000
					2	•3			
					→	→			
						→	→		
							→	→	
					0	•0	0	2	3

This time the 2 in the units column will move three places to the right, as we divide by 10 three times, moving from the units column to the thousandths column. Similarly, the 3 in the tenths column will move three places to the right, to the ten-thousandths column. Because there are no other digits, we need to place zeros in the tenths and hundredths columns to hold the place value of the other digits. In addition, we usually place a zero in the units column for clarity. So 2.3 ÷ 1000 = 0.002 3 .

Next, we shall work out 7.1 × 100 000.

100000	10000	1000	100	10	1	•1/10	1/100	1/1000	1/10000
					7	•1			
				L	L				
			L	L					
		L	L						
	L	L							
L	L								
7	1	0	0	0	0				

This time the 7 in the units column will move five places to the left, as we multiply by 10 five times, moving from the units column to the hundred thousands column. Similarly, the 1 in the tenths column will move five places to the left, to the ten thousands column. Because there are no other digits, we need to place zeros in the thousands, hundreds, tens and units columns to hold the place value of the other digits. So 7.1 × 100 000 = 710 000 .

Finally, we shall work out 7.1×10^{-3}. Now multiplying by 10^{-3} is the same as dividing by 10^3, that is, dividing by 1000.

100000	10000	1000	100	10	1	•1/10	1/100	1/1000	1/10000
					7	•1			
					g	g			
						g	g		
							g	g	
					0	•0	0	7	1

So $7.1 \times 10^{-3} = 0.007\,1$.

Exercises

1. Carry out the following calculations:

(a) 16.21 × 1000 (b) 1.59 × 1000 (c) 0.37 ÷ 100 (d) 27.321 4 ÷ 10 .

3. Decimals and fractions

We shall now look at how decimals and fractions are related.

Decimals are, in fact, decimal fractions. If we put 0.2 on our place value chart, we see that the 2 is in the tenths column. So

$$0.2 = \frac{2}{10} = \frac{1}{5},$$

giving us a direct link between decimals and fractions. Similarly, if we look at 0.25, the 2 represents 2 tenths and the 5 represents 5 hundredths, so

$$0.25 = \frac{2}{10} + \frac{5}{100} = \frac{25}{100} = \frac{1}{4}$$

when written in its lowest form. And if we look at 134.526, this is

$$134 + \frac{5}{10} + \frac{2}{100} + \frac{6}{1000} = 134 + \frac{526}{1000} = 134 + \frac{263}{500} = 134\tfrac{263}{500}.$$

To turn fractions into decimal fractions we simply carry out the division. For example,

$$\frac{1}{2} = 1 \div 2 = 0.5$$

$$\frac{3}{4} = 3 \div 4 = 0.75$$

$$\frac{1}{3} = 1 \div 3 = 0.33333333\ldots = 0.\dot{3}$$

In this case, we can see that the decimal fraction of 1/3 is a non-terminating decimal, and we indicate that the 3 is repeated by writing it with a dot on top, as $\dot{3}$. So 1/3 as a fraction is a more precise notation than a decimal representation which stops after some number of places. The easiest way to convert a fraction to a decimal is to use a calculator to carry out the division, but you should be aware of the inaccuracy involved when the decimal does not terminate.

If you have a mixed fraction, for example $2\frac{5}{6}$, you turn it into a decimal by calculating $5 \div 6$ then adding the 2:

$$2\tfrac{5}{6} = (5 \div 6) + 2 = 0.8\dot{3} + 2 = 2.8\dot{3}.$$

Again in this example we have a non-terminating decimal, and to use $2\frac{5}{6}$ in decimal form we need to approximate, as we shall see in the next section.

Key Point

A decimal is a fraction written in decimal notation.

To find a decimal from a fraction, calculate the numerator divided by the denominator. If the decimal does not terminate and the final digit is repeated indefinitely, this is indicated by a dot placed over that digit, as in $\frac{1}{3} = 0.\dot{3}$. If a whole group of digits is repeated, this is indicated by dots placed over the first and last digits of the group, as in $\frac{1}{11} = 0.\dot{0}\dot{9}$.

Exercises

2. Write the following decimals as fractions:

(a) 0.75 (b) 0.8 (c) 16.275 (d) 6.333333…

3. Write the following fractions as decimals:

(a) $\frac{6}{1}$ (b) $\frac{7}{1}$ (c) $2\frac{12}{4}$.

4. The accuracy of decimals

The accuracy of a number can be given using 'decimal places' or 'significant figures'. We shall look at decimal places first.

Suppose that your calculator gives you an answer of 32.7914, and you are asked to give the answer to one decimal place. This means that you write just 1 digit after the decimal point. So if we put a line after the first decimal place, we can clearly see the other decimal places that will be discarded.

$$32.7|914$$

So our first thought might be that the answer to one decimal place should be 32.7. But before writing our answer down, we need to inspect the decimal place that comes after the line we have drawn. If this digit is a 0, 1, 2, 3 or 4 then the answer required lies to the left of the line, as we had supposed. But if this digit is 5, 6, 7, 8 or 9, as in this example, then the answer is closer to the next tenth up, and the digit to the left of the line must be increased by 1. In this case, 32.79 is closer to 32.8 than it is to 32.7, so to one decimal place the answer is 32.8.

Here are some examples. Write 15.2172 to 3 decimal places (d.p.):

$$15.217|2 \text{ is } 15.217 \text{ to 3 d.p.}$$

Write 0.315 to 2 d.p.:

$$0.31|5 \text{ is } 0.32 \text{ to 2 d.p.}$$

When it comes to dealing with a 5, it is in fact exactly half way between and theoretically the number could be left alone or rounded up. However, mathematical convention is to round up.

Here are some more examples. Round 6.2549 to 1 d.p.:

$$6.2|549 \text{ is } 6.3 \text{ to 1 d.p.}$$

Now round it to 2 d.p.:

$$6.25|49 \text{ is } 6.25 \text{ to 2 d.p.}$$

Now round it to 3 d.p.:

$$6.254|9 \text{ is } 6.255 \text{ to 3 d.p.}$$

To round to significant figures (s.f.), we use the same principles but instead of starting at the decimal point and counting the number of places of decimals required, we start at the left-most digit that is not zero. For example, suppose we are asked to round 27.3721 to 3 s.f. Starting at the first non-zero digit from the left, we count 3 places and draw the line. Looking at the next digit after the line, we decide whether the previous digit stays the same or needs to be increased by 1. So:

$$27.3|721 \text{ is } 27.4 \text{ to 3 s.f.}$$

Round 0.005 214 to 1 s.f.: we start counting from the first non-zero digit on the left, and put the line after 1 digit. So:

$$0.005|214 \text{ is } 0.005 \text{ to 1 s.f.}$$

Write 27 413 200 to 2 s.f.: again start counting from the first non-zero digit on the left and put in the line. But remember, you need to fill in with zeros to ensure that the place values of the 2 and the 7 remain correct.

$$27|413\,200 \text{ is } 27\,000\,000 \text{ to 2 s.f.}$$

Key Point

To round a number, look at the digit that comes after the last place you want. If it is 0, 1, 2, 3 or 4, do nothing. If it is 5, 6, 7, 8 or 9 increase your last digit by one.

Exercises

4. Write these numbers to the given accuracy:

(a)	126.321 to 1 d.p.	(b)	0.579 138 to 1 d.p., and then 3 d.p.	(c)	0.525 to 2 dp
(d)	104 517 to 3 s.f.	(e)	0.000 897 to 1 s.f.	(f)	12.280 5 to 4 s.f.

5. Irrational numbers

Finally, we look briefly at the following question: what are irrational numbers?

Irrational numbers are numbers that cannot be written as proper fractions, that is as a/b where a and b are integers. In decimal fraction form, irrational numbers go on forever, and do not repeat in any pattern. For example

$$\begin{aligned} \pi &= 3.1415926\ldots \\ \sqrt{2} &= 1.414213\ldots \\ \sqrt{3} &= 1.73205\ldots \\ \mathrm{e} &= 2.718281\ldots \end{aligned}$$

So if irrational numbers are in decimal form, they must be approximations to so many significant figures or decimal places.

It is worth noting that recurring decimals such as 0.333333 . . . are not irrational numbers, as they can be written as fractions, eg $1/3$. For these numbers, there is a repeating pattern. An irrational number has no repeating pattern.

Answers

1.

(a) 16 210 (b) 1 590 (c) 0.003 7 (d) 2.732 14

2.

(a) $\frac{3}{4}$ (b) $\frac{4}{5}$ (c) $16\frac{11}{4}$ (d) $6\frac{1}{3}$

3.

(a) 0.6 (b) $0.\dot{6}\dot{3}$ (c) 2.25

4.

(a) 126.3 (b) 0.6, 0.579 (c) 0.53 (d) 105 000 (e) 0.000 9 (f) 12.28

Percentages

In this unit we shall look at the meaning of percentages and carry out calculations involving percentages. We will also look at the use of the percentage button on calculators.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- calculate a percentage of a given quantity;
- increase or decrease a quantity by a given percentage;
- find the original value of a quantity when it has been increased or decreased by a given percentage;
- express one quantity as a percentage of another.

Contents

1. Introduction

The word ‘percentage’ is very familiar to us as it is used regularly in the media to describe anything from changes in the interest rate, to the number of people taking holidays abroad, to the success rate of the latest medical procedures or exam results. Percentages are a useful way of making comparisons, apart from being used to calculate the many taxes that we pay such as VAT, income tax, domestic fuel tax and insurance tax, to name but a few.

So percentages are very much part of our lives. But what does percentage actually mean?

Now ‘per cent’ means ‘out of 100’; and ‘out of’, in mathematical language, means ‘divide by’. So if you score 85% (using the symbol ‘%’ for percentage) on a test then, if there were a possible 100 marks altogether, you would have achieved 85 marks. So

$$85\% = \frac{85}{100}\,.$$

Let us look at some other common percentage amounts, and their fraction and decimal equivalents.

$$\begin{aligned} 75\% &= \tfrac{75}{100} = \tfrac{3}{4} = 0.75 \\ 50\% &= \tfrac{50}{100} = \tfrac{1}{2} = 0.5 \\ 25\% &= \tfrac{25}{100} = \tfrac{1}{4} = 0.25 \\ 10\% &= \tfrac{10}{100} = \tfrac{1}{1} = 0.1 \\ 5\% &= \tfrac{5}{100} = \tfrac{1}{20} = 0.05\,. \end{aligned}$$

It is worth noting that 50% can be found be dividing by 2, and that 10% is easily found by dividing by 10.

Now let us look at writing fractions as percentages. For example, say you get 18 marks out of 20 in a test. What percentage is this?

First, write the information as a fraction. You gained 18 out of 20 marks, so the fraction is $\frac{18}{2}$. Since a percentage requires a denominator of 100, we can turn $\frac{18}{2}$ into a fraction out of 100 by multiplying both numerator and denominator by 5:

$$\frac{18}{20} = \frac{18 \times 5}{20 \times 5} = \frac{90}{100} = 90\%\,.$$

Since we are multiplying both the numerator and the denominator by 5, we are not changing the value of the fraction, merely finding an equivalent fraction.

In that example it was easy to see that, in order to make the denominator 100, we needed to multiply 20 by 5. But if it is not easy to see this, such as with a score of, say, 53 out of 68, then we simply write the amount as a fraction and then multiply by $\frac{100}{100}$:

$$\frac{53}{68} \times \frac{100}{100} = 53 \div 68 \times 100\% = 77.94\%$$

which is 78% to the nearest whole number. Although it is easier to use a calculator for this type of calculation, it is advisable not to use the % button at this stage. We shall look at using the percentage button on a calculator at the end of this unit.

Key Point

Percentage means 'out of 100', which means 'divide by 100'.

To change a fraction to a percentage, divide the numerator by the denominator and multiply by 100%.

Exercises 1

(a) 7 out of every 10 people questioned who expressed a preference liked a certain brand of cereal. What is this as a percentage?

(b) In a test you gained 24 marks out of 40. What percentage is this?

(c) 30 out of 37 gambling sites on the Internet failed to recognise the debit card of a child. What is this as a percentage?

2. Finding percentage amounts

For many calculations, we need to find a certain percentage of a quantity. For example, it is common in some countries to leave a tip of 10% of the cost of your meal for the waiter. Say a meal costs £25.40:

$$10\% \text{ of } £25.40 = \frac{10}{100} \times £25.40 = £2.54\,.$$

As mentioned before, an easy way to find 10% is simply to divide by 10. However the written method shown above is useful for more complicated calculations, such as the commission a salesman earns if he receives 2% of the value of orders he secures. In one month he secures £250,000 worth of orders. How much commission does he receive?

$$2\% \text{ of } £250,000 = \frac{2}{100} \times £250,000 = £5,000\,.$$

Many things that we buy have VAT added to the price, and to calculate the purchase price we have to pay we need to find $17\frac{1}{2}\%$ and add it on to the price. This can be done in two ways.

For example, the cost of a computer is £634 plus VAT. Find the total cost.

$$\begin{aligned} \text{VAT} &= 17\tfrac{1}{2}\% \text{ of } £634 \\ &= \frac{17.5}{100} \times £634 \\ &= £110.95 \\ \text{so total cost} &= £634 + £110.95 \\ &= £744.95\,. \end{aligned}$$

Or, instead of thinking of the total cost as 100% of the price plus $17\frac{1}{2}$% of the price, we can think of it as $117\frac{1}{2}$% of the price, so that

$$117\tfrac{1}{2}\% \text{ of } £634 \quad = \quad \frac{117.5}{100} \times £634 \quad = \quad £744.95\,.$$

Although $17\frac{1}{2}$% seems an awkward percentage to calculate, there is an easy method you can use so that you do not need a calculator. Let us look at the same example again.

		£634	
10%	is	£63.40	(divide by 10)
5%	is	£31.70	(half of 10%)
$2\frac{1}{2}$%	is	£15.85	(half of 5%)
so $17\frac{1}{2}$%	is	£110.95	(add the above).

In a similar way to a percentage increase, there is a percentage decrease. For example, shops often offer discounts on certain goods. A pair of trainers normally costs £75, but they are offered for 10% off in the sale. Find the amount you will pay.

Now 10% of £75 is £7.50, so the sale price is £75 − £7.50 = £67.50.

What you are paying is the 100% of the cost, minus 10% of the cost, so in effect you are paying 90% of the cost. So we could calculate this directly by finding 90% of the cost.

$$90\% \text{ of } £75 \quad = \quad \frac{90}{100} \times £75 \quad = \quad £67.50\,.$$

3. Finding the original amount before a percentage change

Let us look at an example where the price includes VAT, and we need the price excluding VAT.

Example

The cost of a computer is £699 including VAT. Calculate the cost before VAT.

Solution

Now a common mistake here is to take $17\frac{1}{2}$ % of the cost including VAT, and then subtract. But this is wrong, because the VAT is not $17\frac{1}{2}$% of the cost *including* the VAT, which is what we have been given. Instead, the VAT is $17\frac{1}{2}$% of the cost *before* the VAT, and this is what we are trying to find. So we have to use a different method.

Now we have been told that £699 represents the cost including VAT, so that must equal the cost before VAT, plus the VAT itself, which is $17\frac{1}{2}$ % of the cost before VAT. So the total must be 100% + $17\frac{1}{2}$% = $117\frac{1}{2}$% of the cost before VAT. Thus, to find 1% we divide by $117\frac{1}{2}$. So

$$\begin{aligned} 117\tfrac{1}{2}\% \text{ of the price excluding VAT} &= £699, \\ 1\% \text{ of the price excluding VAT} &= \frac{£699}{117.5}\,. \end{aligned}$$

To find the cost before VAT we want 100%, so now we need to multiply by 100. Then

$$\begin{aligned} \text{the price excluding VAT} &= \frac{£699}{117.5} \times 100 \\ &= £594.89\,. \end{aligned}$$

Let us look at another situation where we need to find an original amount before a percentage increase has taken place.

Example

An insurance company charges a customer £320 for his car insurance. The price includes government insurance premium tax at 5%. What is the cost before tax was added?

Solution

Here, the £320 represents 105% of the cost, so to calculate the original cost, 100%, we need to calculate

$$\frac{£320}{105} \times 100 = £304.76\,.$$

Here is one more similar calculation, but this time there has been a reduction in cost.

Example

A shop has reduced the cost of a coat by 15% in a sale, so that the sale price is £127.50. What was the original cost of the coat?

Solution

In this case, £127.50 represents 85% (that is, 100% – 15%) of the original price. So if we write this as a fraction, we divide by 85 to find 1% and then multiply by 100 to find the original price.

$$\frac{£127.50}{85} \times 100 = £150\,.$$

Key Point

If you are given a percentage change and the final amount, write the final amount as 100% plus (or minus) the percentage change, multiplied by the original amount.

4. Expressing a change as a percentage

We might wish to calculate the percentage by which something has increased or decreased. To do this we use the rule

$$\frac{\text{actual increase or decrease}}{\text{original cost}} \times 100\%\,.$$

So you write the amount of change as a fraction of the original amount, and then turn it into a percentage.

Example

Four years ago, a couple paid £180,000 for their house. It is now valued at £350,000. Calculate the percentage increase in the value of the house.

Solution

$$\begin{aligned}
\text{Percentage increase} &= \frac{\text{actual increase}}{\text{original cost}} \times 100\% \\
&= \frac{£350,000 - £180,000}{£180,000} \times 100\% \\
&= \frac{£170,000}{£180,000} \times 100\% \\
&= 94\% \text{ to the nearest } 1\% .
\end{aligned}$$

Let us look at an example where the change has been a decrease.

Example

A car cost £12,000. After 3 years it is worth £8,000. What is the percentage decrease?

Solution

$$\begin{aligned}
\text{Percentage decrease} &= \frac{\text{actual decrease}}{\text{original cost}} \times 100\% \\
&= \frac{£12,000 - £8,000}{£12,000} \times 100\% \\
&= \frac{£4,000}{£12,000} \times 100\% \\
&= 33\% \text{ to the nearest } 1\% .
\end{aligned}$$

Key Point

To write an increase or decrease as a percentage, use the formula

$$\frac{\text{actual increase or decrease}}{\text{original cost}} \times 100\% .$$

5. Calculating percentages using a calculator

Here is a warning about using the percentage button on a calculator: the result depends on when you press the % button in your calculation. Sometimes it has no effect, sometimes it seems to divide by 100, and at other times it multiplies by 100. Here are some examples

- Pressing 48÷ 400% gives an answer of 12. Now 48 400 = 0.12, so pressing the % button has had the effect of multiplying by 100. This has found 48 as a percentage of 400.
- Pressing 1 ÷ 2 × 300% gives the answer 1.5. Now 1 ÷ 2 × 300 = 150, so pressing the % button here has divided by 100. This has found 300% of a half.

- Pressing 400× 50% gives an answer of 200. Now 400 50 = 20, 000, so pressing % here has divided by 100. This has found 50% of 400.
- Pressing 50%× 400 results in 400 on the display, requiring = to be pressed to display an answer of 20,000. So pressing the % button here has had no effect.

Key Point

We recommended that you use the % button on a calculator only when you understand what affect it is having on your calculation.

Exercises 2

(a) What is the amount of VAT (at a rate of $17\frac{1}{2}\%$) which must be paid on an imported computer game costing £16.00?

(b) A visitor to this country buys a souvenir costing £27.50 including VAT at $17\frac{1}{2}\%$. How much VAT can be reclaimed?

(c) At the end of 1999 you bought shares in a company for £100. During 2000 the shares increased in value by 10%. During 2001 the shares decreased in value by 10%. How much were the shares worth at the end of 2001?

(Give your answers to the nearest penny.)

Answers

1.

(a) 70% (b) 60% (c) 81%

2.

(a) £2.80 (b) £4.10 (c) £99.00 .

Ratios

A ratio is a way of comparing two or more similar quantities, by writing two or more numbers separated by colons. The numbers should be whole numbers, and should not include units.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- calculate the ratio of two or more similar quantities, whether or not they are expressed in the same units;
- divide a quantity into a number of parts in given ratios;
- use ratios to scale up, or scale down, a list of ingredients.

Contents

1. Introduction

A ratio is a way of comparing two or more similar quantities. Ratios can be used to compare costs, weights, sizes and other quantities.

For example, suppose we have a model boat which is 1 m long, whereas the actual boat is 25 m long. Then the ratio of the length of the model to the length of the actual boat is 1 to 25. This is written as

$$1 : 25 \,.$$

Note there are no units included, and note also the use of the colon to represent the ratio.

Ratios are also used to describe quantities of different ingredients in mixtures. Pharmacists making up medicines, manufacturers making biscuits and builders making cement all need to make mixtures using ingredients in the correct ratio. If they don't there may be dire consequences! So knowing about ratios is not only very important, but extremely useful and crucial in certain circumstances.

For example, mortar for building a brick wall is made by using 2 parts of cement to 7 parts of sand. Then the ratio of cement to sand is 2 to 7, and is written as

$$2 : 7 \,.$$

2. Simplifying ratios

To make pastry for an apple pie, you need 4 oz flour and 2 oz fat. The ratio of flour to fat is

$$4 : 2 \,.$$

But this ratio can be simplified in the same way that two fractions can be simplified. We just cancel by a common factor. So

$$4 : 2 = 2 : 1 \,.$$

The ratio 2 to 1 is the simplest form of the ratio 4 to 2. And the ratios are equivalent, because the relationship between each pair of numbers is the same.

For example, if we have a ratio 250 to 150, we can simplify it by dividing both numbers by 10 and then by 5 to get 5 to 3:

$$\begin{aligned} 250 &: 150 \\ 25 &: 15 \\ 5 &: 3 \,. \end{aligned}$$

The ratio 5 to 3 is the simplest form of the ratio 250 to 150, and all three ratios are equivalent.

Ratios are normally expressed using whole numbers, so a ratio of 1 to 1.5 would be written as 10 to 15, and then as 2 to 3 in its simplest form:

$$\begin{aligned} 1 &: 1.5 \\ 10 &: 15 \\ 2 &: 3 \,. \end{aligned}$$

Similarly, a ratio $\frac{1}{4}$ to $\frac{5}{8}$ would be written as $\frac{2}{8}$ to $\frac{5}{8}$, and then as 2 to 5 in its simplest form:

$$\frac{1}{4} : \frac{5}{8}$$
$$\frac{2}{8} : \frac{5}{8}$$
$$2 : 5 \, .$$

Now it is very important in a ratio to use the same units for the numbers, as otherwise the ratio will be incorrect and the comparison will be wrong. Take this ratio: 15 pence to £3. The ratio is not 15 to 3 and then 5 to 1. The comparison is wrong. We must have the same units for each number, so we convert them to the same units. It doesn't matter which unit you use, but of course it is just use common sense to choose the unit which gives the simplest numbers. In this case it is obvious that we should use pence, so 15 pence to 300 pence is then simplified to 3 to 60 by dividing by 5. We then simplify it further by dividing by 3 to get 1 to 20. That is the ratio in its simplest form. So

$$15\,\text{p} : \pounds 3$$
$$15 : 3$$
$$5 : 1$$

is wrong, whereas

$$15\,\text{p} : \pounds 3$$
$$15 : 300$$
$$3 : 60$$
$$1 : 20$$

is correct.

Key Point

A ratio is a way of comparing two or more similar quantities. A ratio of 2 cm to 5 cm is written as 2 : 5. A ratio is normally written using whole numbers only, with no units, in its simplest form.

The numbers in a ratio must be written using the same units. If they are not, they should be converted to the same units. It does not matter which units are used for the conversion.

Exercises

1. Express these ratios in their simplest form:

(a)	2 to 10	(b)	80 to 20	(c)	$\frac{1}{3}$ to 1	(d)	50 p : £3.50
(e)	6 m : 30 cm	(f)	1.5 : 1	(g)	10 min : 4 hr	(h)	$\frac{4}{3}$: 3

2. In a class there are 15 girls and 12 boys. What is the ratio of girls to boys?

3. Anna has 75 pence. Rashid has £1.20. What is the ratio of Rasid's money to Anna's money?

3. Using ratios to share quantities

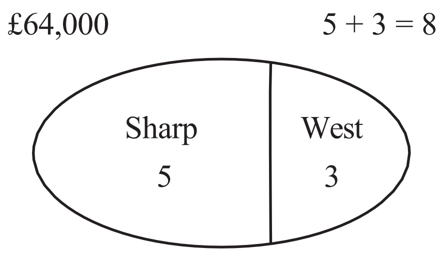
Ratios can be used to share, or divide, quantites of money, weights and so on.

Example

Mrs Sharp and Mr West share an inheritance of £64,000 in the ratio 5 : 3. How much do they each get?

Solution

To calculate the answers we first look at the numbers involved and see the total number of parts into which the inheritance is split. The ratio is 5 to 3. So the total number of parts is 5 plus 3, which is 8.



Now we can work out what one part is worth, and then how much each person gets.

$$\begin{aligned} 1 \text{ part} &= \frac{\pounds 64,000}{8} \\ &= \pounds 8,000. \end{aligned}$$

So Mrs Sharp receives 5 parts, which is $5 \times \pounds 8,000 = \pounds 40,000$ and Mr West receives 3 parts, which is $3 \times \pounds 8,000 = \pounds 24,000$.

We can check our calculations by adding the two amounts together. They should add up to the total value of the inheritance. So

$$\pounds 40,000 + \pounds 24,000 = \pounds 64,000$$

which does equal the original inheritance.

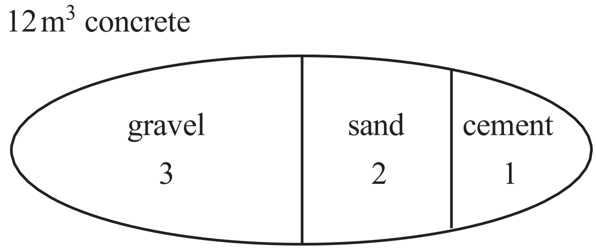
We can also check this calculation in another way. We can work backwards, by taking the two amounts and finding their ratio. The two amounts are both given in the same units, pounds, and so

$$\begin{gathered} 40,000 : 24,000 \\ 40 : 24 \\ 5 : 3 . \end{gathered}$$

Example

Concrete is made by mixing gravel, sand and cement in the ratio 3 : 2 : 1 by volume. How much gravel will be needed to make 12 m^3 of concrete?

Solution



First, we work out the total number of parts into which the concrete is divided: $3 + 2 + 1 = 6$ parts altogether. Using the numbers in the ratio, we know then that gravel makes up 3 parts, sand 2 parts, and cement 1 part. So there are 6 parts altogether, and we have 12 m^3 of concrete, and therefore 1 part must equal 2 m^3. Then as there are 3 parts of gravel, the volume of gravel needed must be $3 \times 2\,m^3$ which is $6\,m^3$:

$$\begin{aligned} 3 + 2 + 1 &= 6 \text{ parts} \\ 6 \text{ parts} &= 12\,m^3 \\ 1 \text{ part} &= \tfrac{12}{6}\,m^3 \\ &= 2\,m^3 \\ \text{so gravel (3 parts)} &= 3 \times 2\,m^3 \\ &= 6\,m^3. \end{aligned}$$

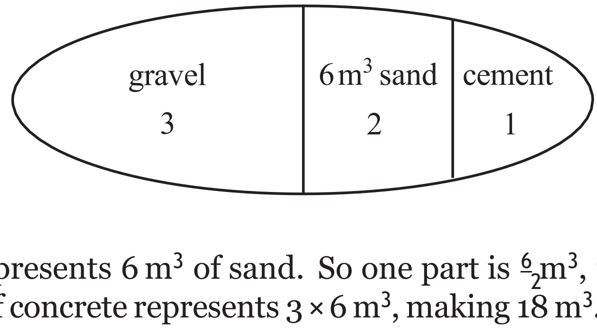
We now need to check the answer. Gravel represents 3 parts out of a total of 6, in other words a half. So half of the total volume of concrete is gravel, and that is half of 12 m^3, which is 6 m^3. So that is indeed the correct answer.

Example

With the same formula for concrete, suppose we have 6 m^3 of sand and an unlimited amount of the other ingredients. How much concrete could we make?

Solution

In this example, the ratio of gravel to sand to cement is still 3 : 2 : 1, so the total number of parts into which the concrete is divided is still $3 + 2 + 1 = 6$. But this time we know the volume of sand, and we have to work out the total volume of concrete that is possible to make.



Two parts of the total represents $6\,m^3$ of sand. So one part is $\frac{6}{2} m^3$, in other words $3\,m^3$, and thus the total of 6 parts of concrete represents $3 \times 6\,m^3$, making 18 m^3. So 18 m^3 of concrete can be made if we have 6 m^3 of sand and an unlimited amount of the other ingredients:

$$\begin{aligned} 3+2+1 &= 6 \text{ parts} \\ 2 \text{ parts} &= 6\,\text{m}^3 \\ 1 \text{ part} &= \tfrac{6}{2}\,\text{m}^3 \\ &= 3\,\text{m}^3 \\ \text{so } 6 \text{ parts} &= 6 \times 3\,\text{m}^3 \\ &= 18\,\text{m}^3\,. \end{aligned}$$

Alternatively, we could have tackled this question by using fractions. Sand represents 2 parts out of a total of 6, which is a third. So if a third of the total is 6 m^3 then the total amount of concrete that could be made would be 3 times 6, giving 18 m^3. This is a good check that our answer is correct.

Example Here is a list of the ingredients to make a quantity of the Greek food houmous sufficient for 6 people.

2 cloves garlic
4 oz chick peas
4 tbs olive oil
5 fl oz tahini paste (houmous for 6 people)

What amounts would be needed so that there will be enough for 9 people?

Solution

The ratio of the amounts is 2 : 4 : 4 : 5 for 6 people. For one person we scale the amounts down, so we divide by 6. Then for 9 people we multiply by 9, and we see after cancelling that we need 3 cloves of garlic, 6oz chick peas, 6 tbs of olive oil, and $7\frac{1}{2}$ fl oz of tahini paste:

$$\begin{array}{ccccccccl} 2 & : & 4 & : & 4 & : & 5 & & \text{(6 people)} \\ \frac{2}{6} & : & \frac{4}{6} & : & \frac{4}{6} & : & \frac{5}{6} & & \text{(1 person)} \\ \frac{1}{3} & : & \frac{2}{3} & : & \frac{2}{3} & : & \frac{5}{6} & & \\ \frac{1}{3}\times 9 & : & \frac{2}{3}\times 9 & : & \frac{2}{3}\times 9 & : & \frac{5}{6}\times 9 & & \text{(1 person)} \\ 3 & : & 6 & : & 6 & : & \frac{15}{2} = 7\frac{1}{2} & & \end{array}$$

giving

3 cloves garlic
6 oz chick peas
6 tbs olive oil
$7\frac{1}{2}$ fl oz tahini paste (houmous for 9 people).

We could have done these calculations more quickly by multiplying each amount by the fraction 9/6, or 3/2 in its simplest form. But it is often safer to work out what the amounts are for one person, and then scale up or down afterwards accordingly.

In conversion problems, it is often better to work out what one of the required amounts represents, and then scale up or down.

Example

If £1 is worth 1.65 euros, what is the value of 50 euros to the nearest penny?

Solution

We are given that 1.65 euros is worth £1 or 100 pence, so 1 euro is worth 100/1.65 pence. Then 50 euros equals 100/1.65 times 50 pence, which is 5000/1.65 pence. Putting this into a calculator gives 3030.3030, which is 3030 pence to the nearest penny, or 30.30. So

$$
\begin{aligned}
1.65 \text{ euros} &= \pounds 1 \\
&= 100 \text{ pence} \\
1 \text{ euro} &= \frac{100}{1.65} \text{ pence} \\
50 \text{ euro} &= \frac{100}{1.65} \times 50 \text{ pence} \\
&= \frac{5000}{1.65} \text{ pence} \\
&= 3030 \text{ pence} \\
&= \pounds 30.30\,.
\end{aligned}
$$

Key Point

When dividing a quanity in a given ratio, it is useful to work out

- the total number of parts into which the quantity is to be divided, and
- the value of one part.

Exercises

4. A map scale is 1 : 20,000. On the map, the distance between two points X and Y is 8.5 cm. What is the actual distance between X and Y?

5. Arminder does a scale drawing of his living room. He uses a scale of 1 : 100. He measures the length of the living room as 13.7 m. How long is the living room on the scale drawing?

6. A recipe to make lasagna for 5 people uses 300 grams of minced beef. How much minced beef would be needed to serve 9 people?

7. The ratio of boys to girls in a youth club is 4 : 5. There are 28 boys. How many girls are there in the youth club?

8. One pound is worth 1.65 euros.

- What is 20 pounds in euros?
- What is 60 euros to the nearest penny?

9. Betty is 12 years old and her sister Liz is 3 years old. Their grandfather gives them £150, which is to be divided between them in the ratio of their ages. How much does each of them get?

10. Divide 360° into three angles in the ratio 1 : 2 : 3.

11. Blue copper sulphate is made from

32 parts of copper
16 parts of sulphur
32 parts of oxygen
48 parts of water

where all the proportions are by weight.

- How much water is there in 5 kg of copper sulphate?
- How much copper sulphate could be made with 96 kg of copper and plenty of all other ingredients?

12. Here are the ingredients for making 18 rock cakes:

9 oz flour
6 oz sugar
6 oz margarine
8 oz mixed dried fruit
2 large eggs.

- Robert wants to make 12 rock cakes. How much margarine does he need?
- Jenny has only 9 oz of sugar and has plenty of all the other ingredients. What is the greatest number of rock cakes she can make?

Answers

1.

(a)	1 : 5	(b)	4 : 1	(c)	1 : 3	(d)	1 : 7	(e)	20 : 1
(f)	3 : 2	(g)	1 : 24	(h)	4 : 9				

2. 5 : 4

3. 8 : 5

4. 1700 m

5. 13.7 cm

6. 540 gm

7. 35

8. (a) 33 euros (b) £36.36, or 3636 pence

9. Betty receives £120, Liz receives £30

10. 60°, 120° and 180°

11. (a) 1875 gm (or 1.875 kg) (b) 384 kg

12. (a) 4 oz (b) 27

Expanding and removing brackets

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In this unit we see how to expand an expression containing brackets. By this we mean to rewrite the expression in an equivalent form without any brackets in. Fluency with this sort of algebraic manipulation is an essential skill which is vital for further study.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that all this becomes second nature. To help you to achieve this, the unit includes a substantial number of such exercises.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- expand expressions involving brackets

Contents

1. Introduction

In this unit we will see how expressions involving brackets can be written in equivalent forms without any brackets. This process is called 'expanding' or 'removing' brackets, and is an important algebraic skill. We motivate the study of this topic by first recalling an investigation that you may have met before at school.

2. Frogs

There is a well-known investigation at GCSE level called 'frogs'. Some of you might well have attempted this particular investigation. We want to start this section by having a look at this investigation because it will show us why we might want to use brackets.

Study Figure 1 which shows three 10 pence coins and three pounds coins.

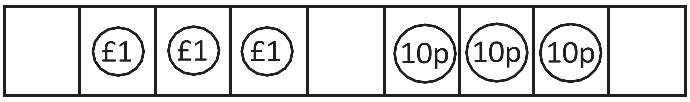

Figure 1.

We want to try and interchange the pound coins on the left side with the 10-pence coins on the right side. We have to do this by using one of two kinds of move. We can either slide into an empty space or we can hop, or jump, over a coin of the opposite kind. We shall do this in stages, and keep count of the number of hops, the number of slides, and the total number of moves.

We can slide the right-most \$1 coin into the vacant space on its right. Then we can hop with the left-most 10p coin into the space vacated by the \$1 coin, as shown in Figure 2.

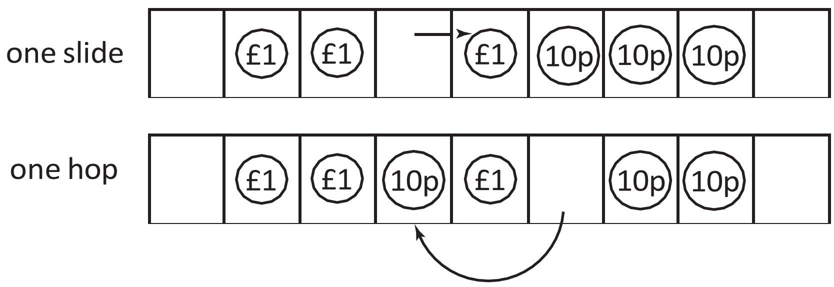

Figure 2.

We now continue in this fashion and at each stage record the number of slides and hops. The details are summarised in Figure 3. If we add up the total numbers of slides and hops we find 6 slides and 9 hops, a total of 15 moves.

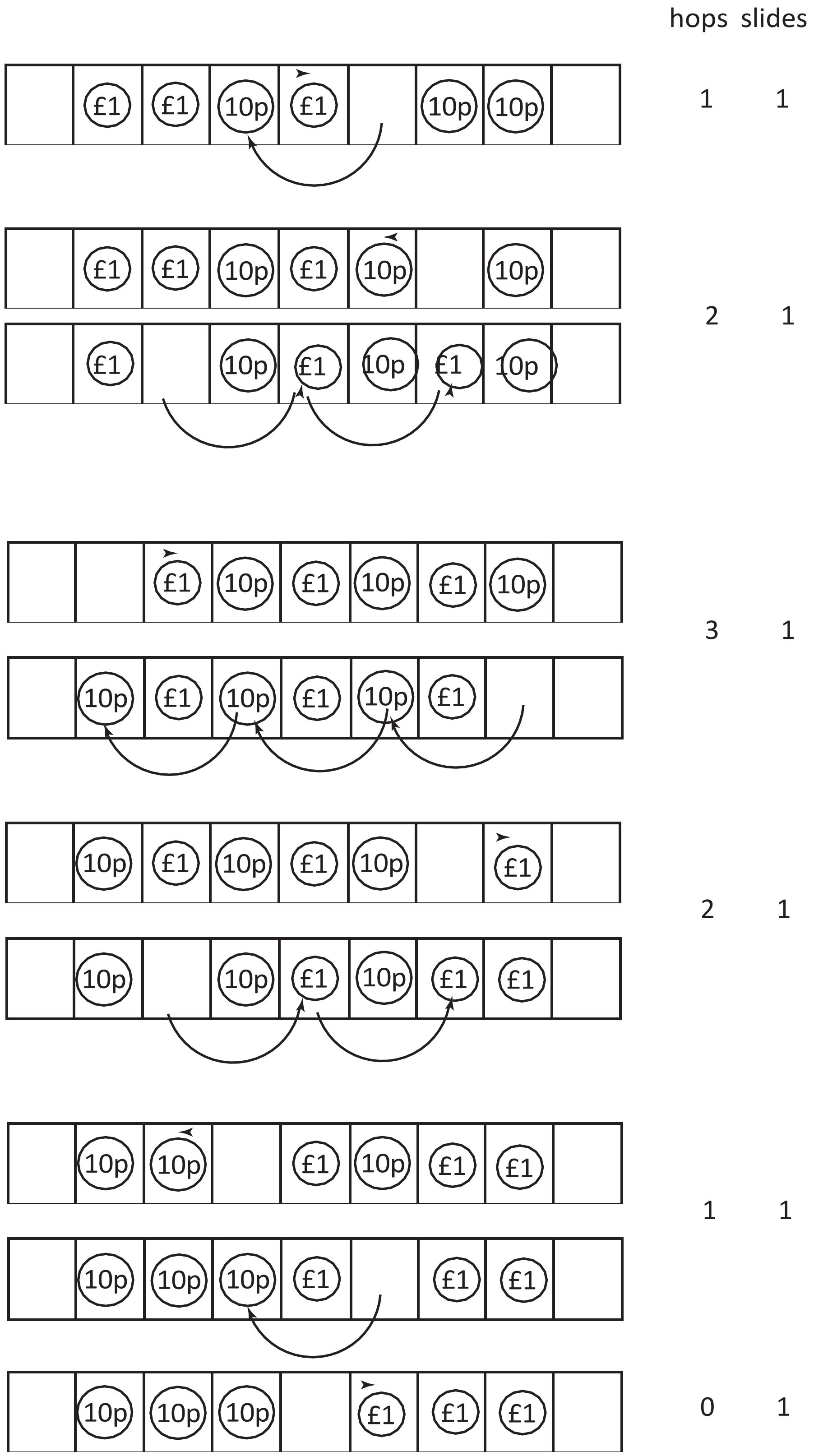

Figure 3.

If we were to repeat this game with a different number of coins on each side we would find the number of slides and hops as given in Table 1.

Table 1

number of coins on each side (n)	Hops	Slides	Moves
1	1	2	3
2	4	4	8
3	9	6	15
4	16	8	24
5	25	10	35

Now the object of most investigations is to try and arrive at a prediction. Can we say what the result would be if we had 10 coins on each side, or 50 coins on each side ? This is the power that mathematics gives us - the power to model in symbols and to be able to use those symbols to make our predictions. Can we make that prediction?

Look at the numbers in the column headed 'Moves'. Notice that $3 = 1 \times 3$, $8 = 2 \times 4$, $15 = 3 \times 5$ and so on. It would appear that the total number of moves is equal to the number of coins, multiplied by the number of coins plus 2.

So if we let the number of coins be n, then the total number of moves appears to be $n \times (n + 2)$, or simply $n(n + 2)$.

From Table 1, notice also that the number of hops appears to be n^2. The number of slides appears to be $2n$. This information is summarised in Table 2.

Table 2

number of coins (n)	Hops	Slides	Moves	
1	$1 = 1^2$	$2 = 2 \times 1$	3	$= 1 \times 3$
2	$4 = 2^2$	$4 = 2 \times 2$	8	$= 2 \times 4$
3	$9 = 3^2$	$6 = 2 \times 3$	15	$= 3 \times 5$
4	$16 = 4^2$	$8 = 2 \times 4$	24	$= 4 \times 6$
5	$25 = 5^2$	$10 = 2 \times 5$	35	$= 5 \times 7$
n	n^2	$2n$	$n(n + 2)$	

Now the sum of the number of hops and the number of slides must equal the total number of moves and so we have

$$n^2 + 2n = n(n + 2)$$

So, this shows us how we can remove brackets. On the right hand side, the n outside must multiply both terms inside the bracket in order to give us the result $n^2 + 2n$.

3. Examples of expanding brackets

Example 1

Expand $3(x+2)$.

The 3 outside must multiply both terms inside the brackets:

$$3(x+2) = 3x+6$$

Example 2

Expand $x(x-y)$.

The x outside must multiply both terms inside the brackets:

$$x(x-y) = x^2 - xy$$

Example 3

Expand $-3a^2(3-b)$.

Both terms inside the brackets must be multiplied by $-3a^2$:

$$-3a^2(3-b) = -9a^2 + 3a^2b$$

Example 4

Expand $-2x(x-y-z)$.

All terms inside the brackets must be multiplied by $-2x$:

$$-2x(x-y-z) = -2x^2 + 2xy + 2xz$$

Key Point

The term outside the brackets multiplies each term inside the brackets.

$$a(b+c) = ab + ac \qquad (b+c)a = ab + ac$$

Exercises

1. Remove the brackets from the following expressions.

   a) $5(x+4)$ b) $2(y-3)$ c) $4(3-a)$ d) $x(2+x)$

   e) $p(q+3)$ f) $-3(2+a)$ g) $s(t-s)$ h) $-2(b-3)$

   i) $5a(2b+3c)$ j) $-y(2x-5y)$ k) $4(x+2y-3z)$ l) $-2a(3a-5b+2c)$

2. Simplify the following expressions.

   a) $5+2(x+1)$ b) $3x+4(2x-1)$ c) $6a-3(a+2)$ d) $x^2-x(1+x)$

   e) $12c-5(2c-1)$ f) $4(y+3)-2y$ g) $5pq-p(2-p)$ h) $7r+2(3+4r)$

   i) $5(2a-b)+3(4b-3a)$

4. Multiplying together two bracketed terms

Let us now have a look at what happens if we want to multiply out expressions where there are two brackets multiplying each other.

Suppose we wish to expand $(x+5)(x+10)$.

We imagine that the term $(x+5)$ is a single quantity and use it to multiply both the x and the 10 in the second pair of brackets:

$$\begin{aligned}(x+5)(x+10) &= (x+5)x + (x+5)10 \\ &= x^2 + 5x + 10x + 50 \\ &= x^2 + 15x + 50\end{aligned}$$

Having seen how to do this, we can shorten the process:

To find $(x+5)(x+10)$:

We must ensure that each term in the first bracket multiplies each term in the second. The arrows in the figure below help us to see that all terms have been taken into account:

$$(x+5)(x+10) = x^2 + 10x + 5x + 50 = x^2 + 15x + 50$$

We can use this process for bracketed expressions like these that lead us to quadratic expressions, and for multiplying pairs of brackets together.

Key Point

When multiplying out two brackets, each containing two terms, we must ensure that each term in the first bracket multiplies each term in the second:

$$(x+a)(x+b) = x^2 + xb + ax + ab = x^2 + (a+b)x + ab$$

Example 5

Expand $(x-7)(x-10)$.

$$\begin{aligned}(x-7)(x-10) &= x^2 - 10x - 7x + 70 \\ &= x^2 - 17x + 70\end{aligned}$$

Example 6

Expand $(x+6)(x-6)$.

$$\begin{aligned}(x+6)(x-6) &= x^2 - 6x + 6x - 36 \\ &= x^2 - 36\end{aligned}$$

Example 7

Expand $(2x-3)(x+1)$.

$$\begin{aligned}(2x-3)(x+1) &= 2x^2 + 2x - 3x - 3 \\ &= 2x^2 - x - 3\end{aligned}$$

Example 8

Expand $(3x-2)(3x+2)$.

$$\begin{aligned}(3x-2)(3x+2) &= 9x^2 + 6x - 6x - 4 \\ &= 9x^2 - 4\end{aligned}$$

Example 9

Expand $(x^2 - 2x^3 + 8)(x+2)$.

$$\begin{aligned}(x^2 - 2x^3 + 8)(x+2) &= x^3 + 2x^2 - 2x^4 - 4x^3 + 8x + 16 \\ &= -2x^4 - 3x^3 + 2x^2 + 8x + 16\end{aligned}$$

Example 10

Expand $(x^2 + x - 2)(x^2 + x - 6)$.

$$\begin{aligned}(x^2 + x - 2)(x^2 + x - 6) &= x^4 + x^3 - 6x^2 + x^3 + x^2 - 6x - 2x^2 - 2x + 12 \\ &= x^4 + 2x^3 - 7x^2 - 8x + 12\end{aligned}$$

Exercises

3. Expand each of the following.

a) $(x+2)(x+3)$ b) $(a+b)(c+3)$ c) $(y-3)(y+2)$

d) $(2x+1)(3x-2)$ e) $(3x-1)(3x+1)$ f) $(5x-1)(x-5)$

g) $(2p+3q)(5p-2q)$ h) $(x+2)(2x^2-x-1)$ i) $(4p+3)(2p-q-5)$

j) $(2z+3)(2z+3)$ k) $(x^2-2x+1)(x^2+4x+3)$ l) $(3x^2-2x+1)(x^2-4x-5)$

5. Dealing with nested brackets

Sometimes we have collections of expressions nested in various sets of brackets.

Example 11

Simplify $a - (b - c) + a + (b - c) + b - (c - a)$.

$$\begin{aligned} a - (b - c) + a + (b - c) + b - (c - a) &= a - b + c + a + b - c + b - c + a \\ &= 3a + b - c \end{aligned}$$

Example 12

In this Example we have a look at some nested brackets.

Simplify $-\{5x - (11y - 3x) - [5y - (3x - 6y)]\}$.

$$\begin{aligned} -\{5x - (11y - 3x) - [5y - (3x - 6y)]\} &= -\{5x - (11y - 3x) - [5y - 3x + 6y]\} \\ &= -\{5x - 11y + 3x - 5y + 3x - 6y\} \\ &= -\{11x - 22y\} \\ &= -11x + 22y \end{aligned}$$

Example 13

Simplify $3b - \{5a - [6a + 2(10a - b)]\}$.

$$\begin{aligned} 3b - \{5a - [6a + 2(10a - b)]\} &= 3b - \{5a - [6a + 20a - 2b]\} \\ &= 3b - \{5a - [26a - 2b]\} \\ &= 3b - \{5a - 26a + 2b\} \\ &= 3b - \{-21a + 2b\} \\ &= 3b + 21a - 2b \\ &= b + 21a \end{aligned}$$

Exercises

4. Simplify the following expressions:

a) $3a - 2(a + b) + 5b + 3(2b - a)$ b) $4x - 2[5y - x + 3(2x - y)]$

Answers

1.

a)	$5x + 20$	b)	$2y - 6$	c)	$12 - 4a$	d)	$2x + x^2$
e)	$pq + 3p$	f)	$-6 - 3a$	g)	$st - s^2$	h)	$-2b + 6$
i)	$10ab + 15ac$	j)	$-2xy + 5y^2$	k)	$4x + 8y - 12z$	l)	$-6a^2 + 10ab - 4ac$

2.
a) $7 + 2x$ b) $11x - 4$ c) $3a - 6$ d) $-x$
e) $2c + 5$ f) $2y + 12$ g) $5pq - 2p + p^2$ h) $15r + 6$
i) $a + 7b$

3.
a) $x^2 + 5x + 6$ b) $ac + 3a + bc + 3b$ c) $y^2 - y - 6$
d) $6x^2 - x - 2$ e) $9x^2 - 1$ f) $5x^2 - 26x + 5$
g) $10p^2 + 11pq - 6q^2$ h) $2x^3 + 3x^2 - 3x - 2$ i) $8p^2 - 4pq - 14p - 3q - 15$
j) $4z^2 + 12z + 9$ k) $x^4 + 2x^3 - 4x^2 - 2x + 3$ l) $3x^4 - 14x^3 - 6x^2 + 6x - 5$

4. a) $-2a + 9b$ b) $-6x - 4y$

Indices or Powers

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A knowledge of powers, or indices as they are often called, is essential for an understanding of most algebraic processes. In this section of text you will learn about powers and rules for manipulating them through a number of worked examples.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- simplify expressions involving indices
- use the rules of indices to simplify expressions involving indices
- use negative and fractional indices.

Contents

1. Introduction

In the section we will be looking at indices or powers. Either name can be used, and both names mean the same thing.

Basically, they are a shorthand way of writing multiplications of the *same* number.
So, suppose we have

$$4 \times 4 \times 4$$

We write this as '4 to the power 3':

$$4^3$$

So

$$4 \times 4 \times 4 = 4^3$$

The number 3 is called the power or index. Note that the plural of index is indices.

Key Point

An index, or power, is used to show that a quantity is repeatedly multiplied by itself.

This can be done with letters as well as numbers. So, we might have:

$$a \times a \times a \times a \times a$$

Since there are five a's multiplied together we write this as 'a to the power 5'.

$$a^5$$

So

$$a \times a \times a \times a \times a = a^5.$$

What if we had $2x^2$ raised to the power 4 ? This means four factors of $2x^2$ multiplied together, that is,

$$2x^2 \times 2x^2 \times 2x^2 \times 2x^2$$

This can be written

$$2 \times 2 \times 2 \times 2 \times x^2 \times x^2 \times x^2 \times x^2$$

which we will see shortly can be written as $16x^8$.

Use of a power or index is simply a form of notation, that is, a way of writing something down. When mathematicians have a way of writing things down they like to use their notation in other ways. For example, what might we mean by

$$a^{-2} \quad \text{or} \quad a^{\frac{1}{2}} \quad \text{or} \quad a^{0} \quad ?$$

To proceed further we need rules to operate with so we can find out what these notations actually mean.

Exercises

1. Evaluate each of the following.

   a) 3^5 b) 7^3 c) 2^9

   d) 5^3 e) 4^4 f) 8^3

2. The first rule

Suppose we have a^3 and we want to multiply it by a^2. That is

$$a^3 \times a^2 \quad = \quad a \times a \times a \quad \times \quad a \times a$$

Altogether there are five a's multiplied together. Clearly, this is the same as a^5. This suggests our first rule.
The first rule tells us that if we are multiplying expressions such as these then we add the indices together. So, if we have

$$a^m \times a^n$$

we add the indices to get

$$a^m \times a^n = a^{m+n}$$

Key Point

$$a^m \times a^n = a^{m+n}$$

3. The second rule

Suppose we had a^4 and we want to raise it all to the power 3. That is

$$(a^4)^3$$

This means

$$a^4 \times a^4 \times a^4$$

Now our first rule tells us that we should add the indices together. So that is

$$a^{12}$$

But note also that 12 is 4×3. This suggests that if we have a^m all raised to the power n the result is obtained by multiplying the two powers to get $a^{m \times n}$, or simply a^{mn}.

Key Point

$$(a^m)^n = a^{mn}$$

4. The third rule

Consider dividing a^7 by a^3.

$$a^7 \div a^3 = \frac{a^7}{a^3} = \frac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a}$$

We can now begin dividing out the common factors of a. Three of the a's at the top and the three a's at the bottom can be divided out, so we are now left with

$$\frac{a^4}{1} \quad \text{that is} \quad a^4$$

The same answer is obtained by subtracting the indices, that is, $7 - 3 = 4$. This suggests our third rule, that $a^m \div a^n = a^{m-n}$.

Key Point

$$a^m \div a^n = a^{m-n}$$

5. What can we do with these rules ? The fourth rule

Let's have a look at a^3 divided by a^3. We know the answer to this. We are dividing a quantity by itself, so the answer has got to be 1.

$$a^3 \div a^3 = 1$$

Let's do this using our rules; rule 3 will help us do this. Rule 3 tells us that to divide the two quantities we subtract the indices:

$$a^3 \div a^3 = a^{3-3} = a^0$$

We appear to have obtained a different answer. We have done the same calculation in two different ways. We have done it correctly in two different ways. So the answers we get, even if they look different, must be the same. So, what we have is $a^0 = 1$.

Key Point

$$a^0 = 1$$

This means that any number raised to the power zero is 1. So

$$2^0 = 1 \qquad (1,000,000)^0 = 1 \qquad \frac{1}{2}^{0} = 1 \qquad (-6)^{0} =$$

However, note that zero itself is an exception to this rule. 0^0 cannot be evaluated. Any number, apart from zero, when raised to the power zero is equal to 1.

6. The fifth rule

Let's have a look now at doing a division again.
Consider a^3 divided by a^7.

$$a^3 \div a^7 = \frac{a^3}{a^7} = \frac{a \times a \times a}{a \times a \times a \times a \times a \times a \times a}$$

Again, we can now begin dividing out the common factors of a. The 3 a's at the top and three of the a's at the bottom can be divided out, so we are now left with

$$a^3 \div a^7 = \frac{1}{a \times a \times a \times a} = \frac{1}{a^4}$$

Now let's use our third rule and do the same calculation by subtracting the indices.

$$a^3 \div a^7 = a^{3-7} = a^{-4}$$

We have done the same calculation in two different ways. We have done it correctly in two different ways. So the answers we get, even if they look different, must be the same. So

$$\frac{1}{a^4} = a^{-4}$$

So a negative sign in the index can be thought of as meaning '1 over'.

Key Point

$$a^{-1} = \frac{1}{a} \qquad \text{and more generally} \qquad a^{-m} = \frac{1}{a^m}$$

Now let's develop this further in the following examples.

In the next two examples we start with an expression which has a negative index, and rewrite it so that it has a positive index, using the rule $a^{-m} = \frac{1}{a^m}$.

Examples

$$2^{-2} = \frac{1}{2^2} = \frac{1}{4} \qquad 5^{-1} = \frac{1}{5^1} = \frac{1}{5}$$

We can reverse the process in order to rewrite quantities so that they have a negative index.

Examples

$$\frac{1}{a} = \frac{1}{a^1} = a^{-1} \qquad \frac{1}{7^2} = 7^{-2}$$

One you should try to remember is $\frac{1}{a} = a^{-1}$ as you will probably use it the most.

But now what about an example like $\frac{1}{7^{-2}}$. Using the Example above, we see that this means $\frac{1}{1/7^2}$. Here we are dividing by a fraction, and to divide by a fraction we need to invert and multiply so:

$$\frac{1}{7^{-2}} = \frac{1}{1/7^2} = 1 \div \frac{1}{7^2} = 1 \times \frac{7^2}{1} = 7^2$$

This illustrates another way of writing the previous keypoint:

Key Point

$$\frac{1}{a^{-m}} = a^m$$

Exercises

2. Evaluate each of the following leaving your answer as a proper fraction.

   a) 2^{-9} b) 3^{-5} c) 4^{-4}

   d) 5^{-3} e) 7^{-3} f) 8^{-3}

7. The sixth rule

So far we have dealt with integer powers both positive and negative. What would we do if we had a fraction for a power, like $a^{\frac{1}{2}}$. To see how to deal with fractional powers consider the following:

Suppose we have two identical numbers multiplying together to give another number, as in, for example

$$7 \times 7 = 49$$

Then we know that 7 is a square root of 49. That is, if

$$7^2 = 49 \qquad \text{then } 7 = \sqrt{49}$$

Now suppose we found that

$$a^p \times a^p = a$$

That is, when we multiplied a^p by itself we got the result a. This means that a^p must be a square root of a.

However, look at this another way: noting that $a = a^1$, and also that, from the first rule, $a^p \times a^p = a^{2p}$ we see that if $a^p \times a^p = a$ then

$$a^{2p} = a^1$$

from which

$$2p = 1$$

and so

$$p = \frac{1}{2}$$

This shows that $a^{1/2}$ must be the square root of a. That is

$$a^{\frac{1}{2}} = \sqrt{a}$$

Key Point

the power 1/2 denotes a square root: $a^{\frac{1}{2}} = \sqrt{a}$

Similarly

$$a^{\frac{1}{3}} = \sqrt[3]{a} \qquad \text{this is the cube root of } a$$

and

$$a^{\frac{1}{4}} = \sqrt[4]{a} \qquad \text{this is the fourth root of } a$$

More generally,

Key Point

$$a^{\frac{1}{q}} = \sqrt[q]{a}$$

Work through the following examples:

Example

What do we mean by $16^{1/4}$?

For this we need to know what number when multiplied together four times gives 16. The answer is 2. So $16^{1/4} = 2$.

Example

What do we mean by $81^{1/2}$? For this we need to know what number when multiplied by itself gives 81. The answer is 9. So $81^{\frac{1}{2}} = \sqrt{81} = 9$.

Example

What about $243^{\frac{1}{5}}$? What number when multiplied together five times gives us 243 ? If we are familiar with times-tables we might spot that $243 = 3 \times 81$, and also that $81 = 9 \times 9$. So

$$243^{1/5} = (3 \times 81)^{1/5} = (3 \times 9 \times 9)^{1/5} = (3 \times 3 \times 3 \times 3 \times 3)^{1/5}$$

So 3 multiplied by itself five times equals 243. Hence

$$243^{1/5} = 3$$

Notice in doing this how important it is to be able to recognise what factors numbers are made up of. For example, it is important to be able to recognise that:

$$16 = 2^4, \qquad 16 = 4^2, \qquad 81 = 9^2, \qquad 81 = 3^4 \qquad \text{and so on.}$$

You will find calculations much easier if you can recognise in numbers their composition as powers of simple numbers such as 2, 3, 4 and 5. Once you have got these firmly fixed in your mind, this sort of calculation becomes straightforward.

Exercises

3. Evaluate each of the following.

   a) $125^{1/3}$ b) $243^{1/5}$ c) $256^{1/4}$

   d) $512^{1/9}$ e) $343^{1/3}$ f) $512^{1/3}$

8. A final result

What happens if we take $a^{\frac{3}{4}}$?

We can write this as follows:

$$a^{\frac{3}{4}} = (a^{\frac{1}{4}})^3 \qquad \text{using the 2nd rule } (a^m)^n = a^{mn}$$

Example

What do we mean by $16^{\frac{3}{4}}$?

$$\begin{aligned} 16^{\frac{3}{4}} &= (16^{\frac{1}{4}})^3 \\ &= (2)^3 \\ &= 8 \end{aligned}$$

We can also think of this calculation performed in a slightly different way. Note that instead of writing $(a^m)^n = a^{mn}$ we could write $(a^n)^m = a^{mn}$ because mn is the same as nm.

Example

What do we mean by $8^{\frac{2}{3}}$? One way of calculating this is to write

$$\begin{aligned} 8^{\frac{2}{3}} &= (8^{\frac{1}{3}})^2 \\ &= (2)^2 \\ &= 4 \end{aligned}$$

Alternatively,

$$\begin{aligned} 8^{\frac{2}{3}} &= (8^2)^{\frac{1}{3}} \\ &= (64)^{\frac{1}{3}} \\ &= 4 \end{aligned}$$

Additional note

Doing this calculation the first way is usually easier as it requires recognising powers of smaller numbers. For example, it is straightforward to evaluate $27^{5/3}$ as

$$27^{5/3} = (27^{1/3})^5 = 3^5 = 243$$

because, at least with practice, you will know that the cube root of 27 is 3. Whereas, evaluation in the following way

$$27^{5/3} = (27^5)^{1/3} = 14348907^{1/3}$$

would require knowledge of the cube root of 14348907.

Writing these results down algebraically we have the following important point:

Key Point

$$a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}} = \sqrt[q]{a^p}$$

$$a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p = (\sqrt[q]{a})^p$$

Both results are exactly the same.

Exercises

4. Evaluate each of the following.
   a) $343^{2/3}$ b) $512^{2/3}$ c) $256^{3/4}$
   d) $125^{4/3}$ e) $512^{7/9}$ f) $243^{6/5}$

5. Evaluate each of the following.
   a) $512^{-7/9}$ b) $243^{-6/5}$ c) $256^{-3/4}$
   d) $125^{-4/3}$ e) $343^{-2/3}$ f) $512^{-2/3}$

9. Further examples

The remainder of this unit provides examples illustrating the use of the rules of indices.

Example

Write $2x^{-\frac{1}{4}}$ using a positive index.

$$2x^{-\frac{1}{4}} = 2 \times \frac{1}{x^{\frac{1}{4}}} = \frac{2}{x^{\frac{1}{4}}}$$

Example

Write $4x^{-2}a^3$ using positive indices.

$$4x^{-2}a^{3} = 4 \times \frac{1}{x^2} \times a^{3} = \frac{4a^3}{x^2}$$

Example

Write $\frac{1}{4a^{-2}}$ using a positive index.

$$\frac{1}{4a^{-2}} = \frac{1}{4} \times \frac{1}{a^{-2}} = \frac{1}{4} \times a^{2} = \frac{a^2}{4}$$

Example

Simplify $a^{-\frac{1}{3}} \times 2a^{-\frac{1}{2}}$.

$$\begin{aligned} a^{-\frac{1}{3}} \times 2a^{-\frac{1}{2}} &= 2a^{-\frac{1}{3}} \times a^{-\frac{1}{2}} \\ &= 2a^{-\frac{5}{6}} \quad \text{adding the indices} \\ &= 2 \times \frac{1}{a^{\frac{5}{6}}} \\ &= \frac{2}{a^{\frac{5}{6}}} \end{aligned}$$

Example

Simplify $\frac{2a^{-2}}{a^{-\frac{3}{2}}}$.

$$\frac{2a^{-2}}{a^{-\frac{3}{2}}} = 2a^{-2} \div a^{-\frac{3}{2}}$$

$$= 2a^{-2-(-3/2)} \quad \text{subtracting the indices}$$

$$= 2a^{-\frac{1}{2}}$$

$$= \frac{2}{a^{\frac{1}{2}}}$$

Example

Simplify $\sqrt[3]{a^2} \times \sqrt[2]{a^3}$.

$$\sqrt[3]{a^2} \times \sqrt[2]{a^3} = a^{\frac{2}{3}} \times a^{\frac{3}{2}}$$

$$= a^{\frac{13}{6}} \quad \text{by adding the indices}$$

Example

Simplify $16^{\frac{3}{4}}$.

$$16^{\frac{3}{4}} = (16^{\frac{1}{4}})^3 = 2^3 = 8$$

Example

Simplify $4^{-\frac{5}{2}}$.

$$4^{-\frac{5}{2}} = \frac{1}{4^{\frac{5}{2}}} = \frac{1}{(4^{\frac{1}{2}})^5} = \frac{1}{2^5} = \frac{1}{32}$$

Example

Simplify $125^{\frac{2}{3}}$.

$$125^{\frac{2}{3}} = (125^{\frac{1}{3}})^2 = 5^2 = 25$$

Example

Simplify $8^{-\frac{2}{3}}$.

$$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{(8^{\frac{1}{3}})^2} = \frac{1}{2^2} = \frac{1}{4}$$

Example

Simplify $\frac{1}{25^{-2}}$.

$$\frac{1}{25^{-2}} = 25^2 = 625$$

Example

Simplify $(243)^{\frac{3}{5}}$.

$$(243)^{\frac{3}{5}} = (243^{\frac{1}{5}})^3 = 3^3 = 27$$

Example

Simplify $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$.

$$\begin{aligned}\left(\frac{81}{16}\right)^{-\frac{3}{4}} &= \frac{1}{\left(\frac{81}{16}\right)^{\frac{3}{4}}} \\ &= \left(\frac{16}{81}\right)^{\frac{3}{4}} \\ &= \left(\left(\frac{16}{81}\right)^{\frac{1}{4}}\right)^3 \\ &= \left(\frac{2}{3}\right)^3 \\ &= \frac{8}{27}\end{aligned}$$

Exercises

6. Evaluate each of the following.

   a) $\left(\frac{4}{9}\right)^2$ b) $\left(\frac{5}{7}\right)^3$ c) $\left(\frac{2}{3}\right)^6$

   d) $\left(\frac{8}{5}\right)^3$ e) $\left(\frac{5}{9}\right)^3$ f) $\left(\frac{4}{3}\right)^4$

7. Evaluate each of the following.

   a) $\left(\frac{4}{9}\right)^{-2}$ b) $\left(\frac{5}{7}\right)^{-3}$ c) $\left(\frac{2}{3}\right)^{-6}$

   d) $\left(\frac{8}{5}\right)^{-3}$ e) $\left(\frac{5}{9}\right)^{-3}$ f) $\left(\frac{4}{3}\right)^{-4}$

8. Evaluate each of the following.

   a) $\left(\frac{32}{243}\right)^{6/5}$ b) $\left(\frac{16}{81}\right)^{3/4}$ c) $\left(\frac{625}{256}\right)^{-1/4}$

   d) $\left(\frac{216}{343}\right)^{1/3}$ e) $\left(\frac{125}{512}\right)^{-2/3}$ f) $\left(\frac{125}{729}\right)^{2/3}$

9. Each of the following expressions can be written as a^n for some value of n. In each case determine the value of n.

   a) $a \times a \times a \times a$ b) $\frac{1}{a \times a \times a}$ c) 1

   d) $\sqrt[3]{a^5}$ e) $a^3 \times a^5$ f) $\frac{a^6}{a}$

   g) $(a^4)^2$ h) $\frac{a^2 \times a^5}{(a^3)^3}$ i) $\sqrt{a} \times \frac{1}{a^{-2}}$

   j) $a^{1/2} \times a^2$ k) $\frac{1}{a^{-3}} \times \frac{1}{a^{-2}}$ l $\frac{1}{(a^{-2})^3}$

10. Simplify each of the following expressions giving your answer in the form Cx^n, where C and n are numbers.

a) $3x^2 \times 2x^4$ b) $5x \times 4x^5$ c) $(2x^3)^4$

d $\frac{8x^6}{2x^3}$ e $\frac{3}{x^2} \times 4x^5$ f) $12x^8 \times \frac{1}{3x^2}$

g) $(5x^3)^{-1}$ h) $(9x^4)^{1/2}$ i) $2x^6 \times \frac{1}{4x^{-2}}$

j) $2x^4 \times \frac{1}{x^5}$ k) $(2x)^4 \times \frac{1}{x^5}$ l) $6x^3 \times \frac{1}{(2x)^{-1}}$

Answers

1. a) 243 b) 343 c) 512
d) 125 e) 256 f) 512

2. a $\frac{1}{512}$ b $\frac{1}{243}$ c $\frac{1}{256}$
d $\frac{1}{125}$ e $\frac{1}{343}$ f) $\frac{1}{512}$

3. a) 5 b) 3 c) 4
d) 2 e) 7 f) 8

4. a) 49 b) 64 c) 64
d) 625 e) 128 f) 729

5. a $\frac{1}{128}$ b $\frac{1}{729}$ c $\frac{1}{64}$
d $\frac{1}{625}$ e $\frac{1}{49}$ f) $\frac{1}{64}$

6. a $\frac{16}{81}$ b $\frac{125}{343}$ c $\frac{64}{729}$
d $\frac{512}{125}$ e $\frac{125}{729}$ f) $\frac{256}{81}$

7. a $\frac{81}{16}$ b $\frac{343}{125}$ c $\frac{729}{64}$
d $\frac{125}{512}$ e $\frac{729}{125}$ f) $\frac{81}{256}$

8. a $\frac{64}{729}$ b $\frac{8}{27}$ c) $\frac{4}{5}$
d) $\frac{6}{7}$ e $\frac{64}{25}$ f) $\frac{25}{81}$

9. a) 4 b) −3 c) 0
d) $\frac{5}{3}$ e) 8 f) 4
g) 8 h) −2 i) $\frac{5}{2}$
j) $\frac{5}{2}$ k) 5 l) 6

10. a) $6x^6$ b) $20x^6$ c) $16x^{12}$
d) $4x^3$ e) $12x^3$ f) $4x^6$
g) $\frac{1}{5}x^{-3}$ h) $3x^2$ i) $\frac{1}{2}x^8$
j) $2x^{-1}$ k) $16x^{-1}$ l) $12x^4$

Surds, and other roots

mc-TY-surds-2009-1

Roots and powers are closely related, but only some roots can be written as whole numbers. Surds are roots which cannot be written in this way. Nevertheless, it is possible to manipulate surds, and to simplify formulæ involving them.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- understand the relationship between negative powers and positive powers;
- understand the relationship between fractional powers and whole-number powers;
- replace formulæ involving roots with formulæ involving fractional powers;
- understand the difference between surds and whole-number roots;
- simplify expressions involving surds;
- rationalise fractions with surds in the denominator.

Contents

1. Introduction

In this unit we are going to explore numbers written as powers, and perform some calculations involving them. In particular, we are going to look at square roots of whole numbers which produce irrational numbers — that is, numbers which cannot be written as fractions. These are called surds.

2. Powers and roots

We know that 2 cubed is $2 \times 2 \times 2$, and we say that we have 2 raised to the power 3, or to the index 3. An easy way of writing this repeated multiplication is by using a 'superscript', so that we would write 2^3:

$$2^3 = 2 \times 2 \times 2 = 8.$$

Similarly, 4 cubed is $4 \times 4 \times 4$, and equals 64. So we write

$$4^3 = 4 \times 4 \times 4 = 64.$$

But what if we have negative powers? What would be the value of 4^{-3}?

To find out, we shall look at what we know already:

$$\begin{aligned} 4^3 &= 4 \times 4 \times 4 &= 64, \\ 4^2 &= 4 \times 4 &= 16, \\ 4^1 &= 4 &= 4, \\ \text{and so } 4^0 &= 4 \div 4 &= 1 \end{aligned}$$

(because to get the answer you divide the previous one by 4).

Now let's continue the pattern:

$$\begin{aligned} 4^{-1} &= 1 \div 4 &= \tfrac{1}{4}, \\ 4^{-2} &= \tfrac{1}{4} \div 4 &= \tfrac{1}{16}, \\ 4^{-3} &= \tfrac{1}{16} \div 4 &= \tfrac{1}{64} \end{aligned}$$

and $\frac{1}{64} = 1/4^3$. So a negative power gives the reciprocal of the number — that is, 1 over the number. Thus $4^{-2} = 1/4^2 = \frac{1}{16}$, and $4^{-1} = 1/4^1 = \frac{1}{4}$. Similarly,

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

and

$$5^{-3} = \frac{1}{5^3} = \frac{1}{125}.$$

A common misconception is that since the power is negative, the result must be negative: as you can see, this is not so.

Now we know that $4^0 = 1$ and $4^1 = 4$, but what is $4^{1/2}$?

Using the rules of indices, we know that $4^{1/2} \times 4^{1/2} = 4^1 = 4$ because $\frac{1}{2} + \frac{1}{2} = 1$. So $4^{1/2}$ equals 2, as $2 \times 2 = 4$. Therefore $4^{1/2}$ is the square root of 4. It is written as $\sqrt{4}$ and equals 2:

$$4^{1/2} = \sqrt{4} = 2.$$

Similarly,

$$9^{1/2} = \sqrt{9} = 3.$$

And in general, any number a raised to the power $\frac{1}{2}$ equals the square root of a:

$$a^{1/2} = \sqrt{a}.$$

So the power, or index, associated with square roots is $\frac{1}{2}$. Also, in the same way that the index $\frac{1}{2}$ represents the square root, other fractions can be used to represent other roots. The cube root of the number 4 is written as

$$4^{1/3} = \sqrt[3]{4}$$

where $\frac{1}{3}$ is the index representing cube root. Similarly, the fourth root of 5 may be written as $5^{1/4} = \sqrt[4]{5}$, and so on. The n-th root is represented by the index $1/n$, and the n-th root of a is written as

$$a^{1/n} = \sqrt[n]{a}.$$

So, for example, if we have $\sqrt[3]{64}$ then this equals 64 to the power $\frac{1}{3}$; and then

$$\begin{aligned} \sqrt[3]{64} &= 64^{1/3} \\ &= (4 \times 4 \times 4)^{1/3} \\ &= 4. \end{aligned}$$

There are some important points about roots, or fractional powers, that we need to remember. First, we can write

$$\begin{aligned} 4^{1/2} \times 4^{1/2} &= 4 \\ \sqrt{4} \times \sqrt{4} &= 4 \\ (\sqrt{4})^2 &= 4 \end{aligned}$$

so that the square root of 4, squared, gives you 4 back again. In fact the square root of any number, squared, gives you that number back again.

Next, if we have a very simple quadratic equation to solve, such as $x^2 = 4$, then the solutions are $x = +2$ or $x = -2$. There are two roots, as $(+2) \times (+2) = 4$ and also $(-2) \times (-2) = 4$. We can write the roots as ± 2. So not all roots are unique. But in a lot of circumstances we only need the positive root, and you do not have to put a plus sign in front of the square root for the positive root. By convention, if there is no sign in front of the square root then the root is taken to be positive.

On the other hand, suppose we were given $\sqrt{-9}$. Could we work this out and get a real answer? Now

$$\sqrt{-9} = (-9)^{1/2},$$

and so we are looking for a number which multiplied by itself gives -9. But there is no such number, because $3 \times 3 = 9$ and also $(-3) \times (-3) = 9$. So you cannot find the square root of a negative number and get a real answer.

Key Point

The square root of a is written as $a^{1/2}$ and is equal to $\sqrt{a}$. The n-th root of a is written as $a^{1/n}$ and is equal to $\sqrt[n]{a}$.

The formula

$$\sqrt{a} \times \sqrt{a} = a$$

can be used to simplify expressions involving square roots.

Not all roots are unique, for example the square root of 4 is 2 or -2, sometimes written as ± 2. But when written without a sign in front, the square root represents the positive root. You cannot find the square root of a negative number.

3. Surds and irrational numbers

We shall now look at some square roots in more detail. Take, for example, $\sqrt{25}$: its value is 5. And the value of $\sqrt{\frac{9}{4}}$ is $\frac{3}{2}$, or $1\frac{1}{2}$. So some square roots can be evaluated as whole numbers or as fractions, in other words as *rational numbers*. But what about $\sqrt{2}$ or $\sqrt{3}$? The roots to these are not whole numbers or fractions, and so they have irrational values. They are usually written as decimals to a given approximation. For example

$$\begin{aligned} \sqrt{2} &= 1.414 && \text{to 3 decimal places,} \\ \sqrt{3} &= 1.732 && \text{to 3 decimal places.} \end{aligned}$$

When we have square roots which give irrational numbers we call them *surds*. So $\sqrt{2}$ and $\sqrt{3}$ are surds. Other surds are

$$\sqrt{5}, \quad \sqrt{6}, \quad \sqrt{7}, \quad \sqrt{8}, \quad \sqrt{10} \qquad \text{and so on.}$$

Surds are often found when using Pythagoras' Theorem, and in trigonometry. So, where possible, it is useful to be able to simplify expressions involving surds. Take, for example, $\sqrt{8}$. This can be written as $\sqrt{4 \times 2}$, which we can rewrite as $\sqrt{4} \times \sqrt{2}$, in other words as $2\sqrt{2}$:

$$\begin{aligned} \sqrt{8} &= \sqrt{4 \times 2} \\ &= \sqrt{4} \times \sqrt{2} \\ &= 2\sqrt{2}. \end{aligned}$$

In general, the square root of a product is the product of the square roots, and vice versa. This is useful to know when simplifying surd expressions.

Now suppose we have been given $\sqrt{5} \times \sqrt{15}$. At first glance this cannot be simplified. But we can rewrite the expression as the square root of 5 times 15, so it is the square root of 75, and 75 can be written as 25 times 5. But 25 is a perfect square, we can use this to simplify the expression.

$$\begin{aligned} \sqrt{5} \times \sqrt{15} &= \sqrt{5 \times 15} \\ &= \sqrt{75} \\ &= \sqrt{25 \times 3} \\ &= \sqrt{25} \times \sqrt{3} \\ &= 5\sqrt{3}. \end{aligned}$$

But watch out if you are given $\sqrt{4+9}$, which is the square root of 13. This does not equal $\sqrt{4} + \sqrt{9}$, which is $2 + 3 = 5$. Now 5 cannot be the answer as that is the square root 25, not the square root of 13.

Key Point

If a positive whole number is not a perfect square, then its square root is called a *surd*. A surd cannot be written as a fraction, and is an example of an irrational number.

4. Simplifying expressions involving surds

Knowing the common square numbers like 4, 9 16, 25, 36 and so on up to 100 is very helpful when simplifying surd expressions, because you know their square roots straight away, and you can use them to simplify more complicated expressions. Suppose we were asked to simplify the expression $\sqrt{400} \times \sqrt{90}$:

$$\begin{aligned} \sqrt{400} \times \sqrt{90} &= \sqrt{4 \times 100} \times \sqrt{9 \times 10} \\ &= \sqrt{4} \times \sqrt{100} \times \sqrt{9} \times \sqrt{10} \\ &= 2 \times 10 \times 3 \times \sqrt{10} \\ &= 60\sqrt{10}, \end{aligned}$$

which cannot be simplified any further.

We can also simplify the expression $\sqrt{2000}/\sqrt{50}$. We get

$$\begin{aligned} \frac{\sqrt{2000}}{\sqrt{50}} &= \sqrt{\frac{2000}{50}} \\ &= \sqrt{40} \\ &= \sqrt{4 \times 10} \\ &= \sqrt{4} \times \sqrt{10} \\ &= 2\sqrt{10}. \end{aligned}$$

Key Point

The following formulæ may be used to simplify expressions involving surds:

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$
$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}\,.$$

If we have a product of brackets involving surds, for example $(1+\sqrt{3})(2-\sqrt{2})$, we can expand out the brackets in the usual way:

$$\begin{aligned}(1+\sqrt{3})(2-\sqrt{2}) &= 2-\sqrt{2}+2\sqrt{3}-\sqrt{3}\sqrt{2} \\ &= 2-\sqrt{2}+2\sqrt{3}-\sqrt{6}\,.\end{aligned}$$

But what if we have this expression, $(1+\sqrt{3})(1-\sqrt{3})$? If we expand this out and simplify the answer, we get

$$(1+\sqrt{3})(1-\sqrt{3}) = 1-\sqrt{3}+\sqrt{3}-\sqrt{3}\sqrt{3} = 1-\sqrt{9} = 1-3 = -2\,.$$

So the product $(1+\sqrt{3})(1-\sqrt{3})$ does not involve surds at all. This is an example of a general result known as the *difference of two squares*. This general result may be written as

$$a^2-b^2 = (a+b)(a-b)$$

for any numbers a and b. In our example $a=1$ and $b=\sqrt{3}$, so

$$(1+\sqrt{3})(1-\sqrt{3}) = 1^2-(\sqrt{3})^2 = 1-3 = -2\,.$$

The expansion of the difference of two squares is another useful fact to know and remember.

Key Point

The formula for the difference of two squares is

$$a^2-b^2 = (a+b)(a-b)\,.$$

5. Rationalising expressions containing surds

Sometimes in calculations we obtain surds as denominators, for example $1/\sqrt{13}$. It is best not to give surd answers in this way. Instead, we use a technique called *rationalisation*. This changes the surd denominator, which is irrational, into a whole number.

To see how to do this, take our example $1/\sqrt{13}$. To rationalise this, we multiply by the fraction $\sqrt{13}/\sqrt{13}$ which is equal to 1. When we multiply by this fraction we do not change the value of our original expression. We obtain

$$\begin{aligned} \frac{1}{\sqrt{13}} &= \frac{1}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}} \\ &= \frac{\sqrt{13}}{13}. \end{aligned}$$

We can also use the expansion of the difference of two squares to rationalise more complicated expressions involving surds.

Example

Rationalise the expression

$$\frac{1}{1+\sqrt{2}}.$$

Solution

If we multiply this expression by this fraction $(1-\sqrt{2})/(1-\sqrt{2})$ we do not change its value, as the new fraction is equal to 1. When we do this, we use the formula for the difference of two squares to work out

$$(1+\sqrt{2}) \times (1-\sqrt{2}),$$

and that gives us $1^2 - (\sqrt{2})^2$, which is $1-2=-1$. So now we have a whole number in the denominator of our fraction, and we can divide through. We get

$$\begin{aligned} \frac{1}{1+\sqrt{2}} &= \frac{1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} \\ &= \frac{1-\sqrt{2}}{1^2-(\sqrt{2})^2} \\ &= \frac{1-\sqrt{2}}{-1} \\ &= -1+\sqrt{2} \\ &= \sqrt{2}-1. \end{aligned}$$

Example

Rationalise the expression

$$\frac{1}{\sqrt{5}-\sqrt{3}}.$$

Solution

To rationalise this, we must multiply it by a fraction which equals 1. To choose a suitable fraction, we think of the difference of two squares. As our expression has $\sqrt{5}-\sqrt{3}$ in the denominator, the fraction to use is $(\sqrt{5}+\sqrt{3})$ over $(\sqrt{5}+\sqrt{3})$:

$$\begin{aligned}
\frac{1}{\sqrt{5}-\sqrt{3}} &= \frac{1}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} \\
&= \frac{\sqrt{5}+\sqrt{3}}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})} \\
&= \frac{\sqrt{5}+\sqrt{3}}{(\sqrt{5})^2-(\sqrt{3})^2} \\
&= \frac{\sqrt{5}+\sqrt{3}}{2} \\
&= \frac{\sqrt{5}}{2}+\frac{\sqrt{3}}{2}.
\end{aligned}$$

Key Point

A surd expression in the form

$$\frac{1}{(\sqrt{a}+\sqrt{b})} \quad \text{or} \quad \frac{1}{(\sqrt{a}-\sqrt{b})}$$

can be written as $c\sqrt{a}+d\sqrt{b}$ by using the formula for the difference of two squares.

Exercises

1. Simplify the following expressions:

(a) $\sqrt{40}\times\sqrt{200}$ (b) $\sqrt{90}\times\sqrt{600000}$ (c) $\sqrt{1000}/\sqrt{40}$ (d) $(1-\sqrt{5})(1+\sqrt{5})$

(e) $(\sqrt{7}+3)(\sqrt{7}-3)$

2. Rationalise the following expressions:

(a) $1/\sqrt{11}$ (b) $30/\sqrt{3}$ (c) $1/(1-\sqrt{3})$ (d) $1/(\sqrt{7}+\sqrt{2})$ (e) $2/(\sqrt{5}-\sqrt{7})$

Answers

1. (a) $40\sqrt{5}$ (b) $3000\sqrt{6}$ (c) 5 (d) -4 (e) -2

2. (a) $\sqrt{11}/11$ (b) $10\sqrt{3}$ (c) $-\frac{1}{2}-\sqrt{3}/2$ (d) $\sqrt{7}/5-\sqrt{2}/5$ (e) $-\sqrt{5}-\sqrt{7}$

Transposition of formulae

mc-TY-transposition-2009-1

In mathematics, engineering and science, formulae are used to relate physical quantities to each other. They provide rules so that if we know the values of certain quantities, we can calculate the values of others. In this unit we discuss how formulae can be transposed, or transformed, or rearranged.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- transpose formulae in order to make other variables the subject of the formula

Contents

1. Introduction

Consider the formula for the period, T, of a simple pendulum of length l:

$$T = 2\pi\sqrt{\frac{l}{g}}, \qquad \text{where } l \text{ is the length of the pendulum}$$

Now, on Earth, we tend to regard g, the acceleration due to gravity, as being fixed. It varies a little with altitude but for most purposes we can regard it as a constant.

Just suppose we had a pendulum of a fixed length l, and we took it somewhere else, say the moon, or Mars. The gravity there would not be the same and so the value of g would be different from its value on Earth.

Suppose we wanted to measure g. One way would be to take the pendulum, set it swinging and measure the time, T , for one complete cycle. Once we have measured the time, we could then use that to calculate g. But before we could do this, we would need to know what g was, in terms of the rest of the symbols in the formula. So we need to re-arrange the formula so that it states "$g =?$". We will show you how to do this rearrangement later in the video.

To rearrange, transform, or tranpose the formula, we need many of the techniques used to solve equations. So, the video '*Solving Linear Equations in One Variable*' might be very useful to have a look at.

2. Solving a simple linear equation

Before we look at rearranging more complicated formulae we recap by having a look at a simple linear equation. Suppose we wanted to solve

$$3x + 5 = 6 - 3(5 - 2x)$$

Our aim is to end up with an expression for x, that is '$x =?$'. We start by expanding the brackets on the right.

$$\begin{aligned} 3x + 5 &= 6 - 15 + 6x \\ 3x + 5 &= 6x - 9 \end{aligned}$$

We then proceed to manipulate this to try to get all the terms involving x on to one side. We must preserve the balance in the original equation by doing exactly the same operations to both sides.

Subtracting $3x$ from both sides:

$$\begin{aligned} 3x - 3x + 5 &= 6x - 3x - 9 \\ 5 &= 3x - 9 \end{aligned}$$

Adding 9 to both sides:

$$\begin{aligned} 5 + 9 &= 3x - 9 + 9 \\ 14 &= 3x \end{aligned}$$

and so

$$x = \frac{14}{3}$$

We have obtained an expression for x as required.

With practice the amount of working written down will reduce because you will be able to carry out many of the stages at the same time. Having gone through the steps of solving an equation, the same technique, particularly the idea of keeping the balance by *doing the same thing to both sides*, is what we are going to do when we look at the transformation of formulae.

3. Transposition of simple formulae

Example

Consider the formula $v = u + at$. Suppose we wish to transpose this formula to obtain one for t.

Because we want to obtain t on its own we start by subtracting u from each side:

$$\begin{aligned} v &= u + at \\ v - u &= at \end{aligned}$$

We now divide everything on both sides by a.

$$\frac{v-u}{a} = \frac{at}{a} = t$$

and so finally $t = \dfrac{v-u}{a}$. We have transposed the formula to find an expression for t.

Example

Consider the formula $v^2 = u^2 + 2as$ and suppose we wish to transpose it to find u.

We want to obtain u on its own and so we begin by subtracting $2as$ from each side.

$$\begin{aligned} v^2 &= u^2 + 2as \\ v^2 - 2as &= u^2 \end{aligned}$$

Finally, taking the square root of both sides:

$$u = \sqrt{v^2 - 2as}$$

Notice we need to take the square root of the whole term ($\sqrt{v^2 - 2as}$) in order to find u.

Example

Consider the formula $s = ut + \frac{1}{2}at^2$. Suppose we want to transpose it to find a.

Because we want a on its own, we begin by subtracting ut from both sides.

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ s - ut &= \frac{1}{2}at^2 \end{aligned}$$

Multiplying both sides by 2:

$$2(s - ut) = at^2$$

Dividing both sides by t^2:

$$\frac{2(s-ut)}{t^2} = a$$

and so

$$a = \frac{2(s-ut)}{t^2}$$

Example

Suppose we wish to rearrange $y(2x+1) = x+1$ in order to find x.

Notice that x occurs both on the left and on the right. We need to try to get all the terms involving x together. We begin by expanding the brackets on the left:

$$\begin{aligned} y(2x+1) &= x+1 \\ 2xy + y &= x+1 \end{aligned}$$

Subtracting x from both sides:

$$2xy - x + y = 1$$

The left-hand side now has two terms involving x. We can factorise these as follows:

$$x(2y-1) + y = 1$$

Then subtracting y from both sides:

$$x(2y-1) = 1 - y$$

and finally, dividing both sides by $(2y-1)$

$$x = \frac{1-y}{2y-1}$$

Example

Suppose we wish to rearrange $\frac{y}{y+x} + 5 = x$ to find an expression for y.

We begin by multiplying *every term* on both sides by $(y+x)$ in order to remove the fractions:

$$y + 5(y+x) = x(y+x)$$

Next we multiply out the brackets:

$$y + 5y + 5x = xy + x^2$$

We try to get all the terms involving y onto the left-hand side. Subtracting xy from both sides:

$$6y - xy + 5x = x^2$$

Subtracting $5x$ from both sides, and taking out the common factor y we have

$$y(6-x) = x^2 - 5x$$

Finally, dividing both sides by $6-x$ we obtain

$$y = \frac{x^2 - 5x}{(6-x)}$$

Exercise 1

Rearrange each of the following formulae to make the quantity shown the subject.

1. $v = u + at, \quad u$
2. $v^2 = u^2 + 2as, \quad s$
3. $s = vt - \frac{1}{2}at^2, \quad a$
4. $p = 2(w + h), \quad h$
5. $A = 2\pi r^2 + 2\pi rh, \quad h$
6. $E = \frac{1}{2}mv^2 + mgh, \quad v$
7. $E = \frac{1}{2}mv^2 + mgh, \quad m$
8. $a(3b - 1) = 2b + 2, \quad b$
9. $\frac{t}{2t - s} = 3s, \quad t$
10. $\frac{s}{2t - s} + 5 = 3t, \quad s$

4. The formula for the simple pendulum

We began with the formula $T = 2\pi\sqrt{\frac{l}{g}}$. Let us now try to rearrange this to find an expression for g.

We begin by squaring both sides of the equation in order to remove the square root.

$$T^2 = (2\pi)^2 \frac{l}{g}$$

To remove the fraction we multiply both sides by g:

$$T^2 g = (2\pi)^2 l$$

Dividing both sides by T^2 gives

$$g = \frac{(2\pi)^2 l}{T^2}$$

By observing the two square terms on the right, we note that this formula could be written, if we wish, in the equivalent form

$$g = \left(\frac{2\pi}{T}\right)^2 l$$

5. Further examples of useful formulae

Example - the lens formula

The so-called lens formula, which is used in optics, is given by

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

Suppose we want to rearrange this formula to find u.

Because we want to isolate u we begin by subtracting $\frac{1}{v}$ from both sides.

$$\frac{1}{f} - \frac{1}{v} = \frac{1}{u}$$

The left-hand side fractions can be combined by expressing them over a common denominator

$$\frac{v-f}{fv} = \frac{1}{u}$$

Inverting both sides

$$\frac{fv}{v-f} = u$$

and so $u = \frac{fv}{v-f}$ as required.

Example

The formula $T = \frac{T_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$ arises in the study of relativity. Suppose we want to rearrange it to find an expression for $\frac{v}{c}$.

We begin by noticing that if we square both sides this will remove the square root term (i.e. the power $\frac{1}{2}$) on the right-hand side. So squaring:

$$T^2 = \frac{T_0^2}{1 - \frac{v^2}{c^2}}$$

We remove the fraction by multiplying both sides by $1 - \frac{v^2}{c^2}$:

$$T^2\left(1 - \frac{v^2}{c^2}\right) = T_0^2$$

Dividing both sides by T^2:

$$1 - \frac{v^2}{c^2} = \frac{T_0^2}{T^2}$$

Adding $\frac{v^2}{c^2}$ to both sides gives

$$1 = \frac{T_0^2}{T^2} + \frac{v^2}{c^2}$$

Subtracting $\frac{T_0^2}{T^2}$ from both sides:

$$1-\frac{T_0^2}{T^2}=\frac{v^2}{c^2}$$

Finally, taking the square root of both sides

$$\frac{v}{c}=\sqrt{1-\frac{T_0^2}{T^2}}$$

as required.

Exercise 2

Rearrange each of the following formulae to make the quantity shown the subject.

1. $y=a+\frac{1}{x}, \quad x$
2. $y=a+\frac{1}{1-x}, \quad x$
3. $P=\frac{P_0}{1-r^2}, \quad r$
4. $m=k\sqrt{a(1-x)}, \quad x$
5. $V=\frac{V_0}{\sqrt{r^2-1}}, \quad r$

Answers

Exercise 1

1. $u=v-at$ 2. $s=\frac{v^2-u^2}{2a}$ 3. $a=\frac{2(vt-s)}{t^2}$ 4. $h=\frac{1}{2}(p-2w)$
5. $h=\frac{A-2\pi r^2}{2\pi r}$ 6. $v=\sqrt{\frac{2(E-mgh)}{m}}$ 7. $m=\frac{2E}{v^2+2gh}$ 8. $b=\frac{2+a}{3a-2}$
9. $t=\frac{3s^2}{6s-1}$ 10. $s=\frac{6t^2-10t}{3t-4}$.

Exercise 2

1. $x=\frac{1}{y-a}$ 2. $x=1-\frac{1}{y-a}$ 3. $r=\sqrt{1-\frac{P_0}{P}}$
4. $x=1-\frac{1}{a}\left(\frac{m}{k}\right)^2$ 5. $r=\sqrt{1+\left(\frac{V_0}{V}\right)^2}$

Linear equations in one variable

mc-TY-simplelinear-2009-1

In this unit we give examples of simple linear equations and show you how these can be solved. In any equation there is an unknown quantity, x say, that we are trying to find. In a linear equation this unknown quantity will appear only as a multiple of x, and not as a function of x such as x^2, x^3, $\sqrt{x}$, $\sin x$ and so on. Linear equations occur so frequently in the solution of other problems that a thorough understanding of them is essential.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that all this becomes second nature. To help you to achieve this, the unit includes a substantial number of such exercises.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- recognise simple linear equations
- solve simple linear equations
- check that your solutions are correct by substitution

Contents

1. Introduction

In this unit we are going to be looking at simple equations in one variable, and the equations will be linear - that means there'll be no x^2 terms and no x^3's, just x's and numbers. For example, we will see how to solve the equation $3x + 15 = x + 25$.

2. Solving equations by collecting terms

Suppose we wish to solve the equation

$$3x + 15 = x + 25$$

The important thing to remember about any equation is that the equals sign represents a balance. What an equals sign says is that what's on the left-hand side is exactly the same as what's on the right-hand side. So, if we do anything to one side of the equation we have to do it to the other side. If we don't, the balance is disturbed. Therefore, whatever operation we perform on either side of the equation, so long as it's done in exactly the same way on each side the balance will be preserved.

Our first step in solving any equation is to attempt to gather all the x's together and to gather all the numbers together.

From

$$3x + 15 = x + 25$$

we can subtract x from each side, because this will remove it entirely from the right, to give

$$2x + 15 = 25$$

We can subtract 15 from each side to give

$$2x = 10$$

and finally, by dividing each side by 2 we obtain

$$x = 5$$

So the solution of the equation is $x = 5$. This solution should be checked by substitution into the original equation in order to check that both sides are the same. If we do this, the left is $3(5) + 15 = 30$. The right is $5 + 25 = 30$. So the left equals the right and we have checked that the solution is correct.

Example

Solve the equation $2x + 3 = 6 - (2x - 3)$.

Solution

From $2x + 3 = 6 - (2x - 3)$ we first remove the brackets on the right to give

$$2x + 3 = 6 - 2x + 3$$

so that

$$2x + 3 = 9 - 2x$$

We are now in the same position as we were in during the first Example. We need to get the x's together by adding $2x$ to each side.

$$4x + 3 = 9$$

Now take 3 away from each side:

$$4x = 6$$

so that

$$\begin{aligned} x &= \frac{6}{4} \quad \text{(by dividing both sides by 4)} \\ &= \frac{3}{2} \\ &= 1\frac{1}{2} \end{aligned}$$

When solving simple equations we should always check the solution by taking our answer and substituting it in the original equation to check that the left- and right- hand sides are the same.

Substituting $x = 1\frac{1}{2}$ in the left-hand side gives:

$$2 \times \left(1\frac{1}{2}\right) + 3 = 3 + 3 = 6$$

Substituting $x = 1\frac{1}{2}$ in the right-hand side gives:

$$6 - \left(2 \times \left(1\frac{1}{2}\right) - 3\right) = 6 - 0 = 6$$

So again, the left- and right- hand sides are equal - we've got that balance, so we know that we've got the right answer.

Exercises

1. Solve the following equations.

a)	$x + 5 = 9$	b)	$12 - x = 7$	c)	$5x = 3$
d)	$4x + 10 = 2$	e)	$5 - 3x = -4$	f)	$2 + 14x = 30$
g)	$9 + 5x = 3x + 13$	h)	$4 - 3x = 8 + x$	i)	$5 + 3(x - 1) = 5x - 6$

3. Solving equations by removing brackets & collecting terms

Example

Solve the equation

$$8(x - 3) - (6 - 2x) = 2(x + 2) - 5(5 - x)$$

We begin by multiplying out the brackets, taking care, in particular, with any minus signs.

$$8x - 24 - 6 + 2x = 2x + 4 - 25 + 5x$$

Each side can be tidied up by collecting the x terms and the numbers together.

$$10x - 30 = 7x - 21$$

Now take $7x$ from each side, and then add 30 to each side:

$$\begin{aligned} 3x - 30 &= -21 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

And again you should take the solution $(x = 3)$, substitute it back into the original equation to check that we have got the correct answer. On the left:

$$8(x - 3) - (6 - 2x) = 8(3 - 3) - (6 - 2(3)) = 0 - 0 = 0.$$

On the right:

$$2(x + 2) - 5(5 - x) = 2(3 + 2) - 5(5 - 3) = 10 - 10 = 0.$$

So both sides equal zero. The equation balances and so $x = 3$ is the solution.

Example

Solve the equation

$$(x + 1)(2x + 1) = (x + 3)(2x + 3) - 14$$

We begin by removing the brackets.

$$2x^2 + x + 2x + 1 = 2x^2 + 3x + 6x + 9 - 14$$

So

$$2x^2 + 3x + 1 = 2x^2 + 9x - 5$$

Remember we stated that we are dealing in this unit with linear equations, so there should be no x^2 terms. In fact, they all cancel out:

There is a term $2x^2$ on both sides. We can subtract $2x^2$ from both sides to leave

$$3x + 1 = 9x - 5$$

We can now proceed as in the earlier examples.

$$\begin{aligned} 3x + 1 &= 9x - 5 \\ 1 &= 6x - 5 \\ 6 &= 6x \\ 1 &= x \end{aligned}$$

So the solution is $x = 1$. As before, we can substitute it back into the original equation as a check.

On the left:

$$(x + 1)(2x + 1) = (1 + 1)(2 + 1) = (2)(3) = 6$$

On the right

$$(x + 3)(2x + 3) - 14 = (1 + 3)(2 + 3) - 14 = (4)(5) - 14 = 20 - 14 = 6$$

So both sides equal 6 and the equations balance when $x = 1$. The solution is $x = 1$.

Exercises

2. Solve the following equations.

a) $5(3-x)-2(4-3x)=11-2(x-1)$ b) $6-4(x+3)=2(x-1)$

c) $5(1-2x)+2(3-x)=3(x+4)+14$

4. Linear equations with fractional coefficients

Example

Solve the equation

$$\frac{4(x+2)}{5}=7+\frac{5x}{13}$$

Solution

In this Example the fractions are the cause of the difficulty. We want to try to remove them and work with whole numbers. Multiplying both sides by 5 and then by 13 will remove the fractions. This is equivalent to multiplying both sides by the lowest common denominator, which is $5\times 13=65$.

$$\begin{aligned}
\frac{4(x+2)}{5} &= 7+\frac{5x}{13} \\
65\times\frac{4(x+2)}{5} &= 65\left(7+\frac{5x}{13}\right) \\
65\times\frac{4(x+2)}{5} &= 65\times 7 \quad +65\times\frac{5x}{13} \\
{}^{13}\cancel{65}\times\frac{4(x+2)}{\cancel{5}_1} &= 65\times 7 \quad +{}^{5}\cancel{65}\times\frac{5x}{\cancel{13}_1} \\
52(x+2) &= 455+25x
\end{aligned}$$

This is a much more familiar form, like the earlier examples. Multiply out the brackets, collect together x terms and collect together the numbers.

$$\begin{aligned}
52x+104 &= 455+25x \\
27x &= 351 \\
x &= \frac{351}{27} \\
&= 13
\end{aligned}$$

We should go back and check this solution to make sure it is correct. So let's do that.
On the left hand side:

$$\frac{4(x+2)}{5}=\frac{4(13+2)}{5}=\frac{60}{5}=12$$

On the right:

$$7+\frac{5x}{13}=7+\frac{(5)(13)}{13}=7+5=12$$

We see that the left and right sides are equal. So the solution $x=13$ is correct.

Example

Solve $\frac{x+5}{6} - \frac{x+1}{9} = \frac{x+3}{4}$

Solution

In this example there are no brackets. Does that make any difference? The thing you have to remember is that a division line also acts as a bracket. For example, $\frac{x+5}{6}$ means that all of $x+5$ is divided by 6. So it is helpful to put brackets around these terms.

$$\frac{(x+5)}{6} - \frac{(x+1)}{9} = \frac{(x+3)}{4}$$

Now we need a common denominator; we need a number into which all of the individual denominators (6, 9 and 4) will divide exactly. The lowest number into which they all divide is 36. So let's multiply throughout by 36.

$$\frac{36(x+5)}{6} - \frac{36(x+1)}{9} = \frac{36(x+3)}{4}$$

Notice that we've made it quite clear by using the brackets what the 36 is multiplying. Each term can be simplifed by dividing top and bottom by the common factors.

$$\frac{{}^{6}\cancel{36}(x+5)}{6_1} - \frac{{}^{4}\cancel{36}(x+1)}{9_1} = \frac{{}^{9}\cancel{36}(x+3)}{4_1}$$

$$\frac{6(x+5)}{1} - \frac{4(x+1)}{1} = \frac{9(x+3)}{1}$$

from which

$$6x + 30 - 4x - 4 = 9x + 27$$

Simplifying the left and right hand sides separately

$$2x + 26 = 9x + 27$$

Then take $2x$ away from each side to give

$$26 = 7x + 27$$

Take 27 away from each side

$$-1 = 7x$$

and finally

$$x = -\frac{1}{7}$$

So, the solution is $x = -\frac{1}{7}$. This should be checked by substitution into the original equation as follows:

Substitution of $x = -\frac{1}{7}$ into the left-hand side of the original equation we find:

$$\frac{-\frac{1}{7}+5}{6} - \frac{-\frac{1}{7}+1}{9}$$

which simplifies as follows:

$$\frac{\frac{-1+35}{7}}{6} - \frac{\frac{-1+7}{7}}{9}$$

and further to

$$\frac{34}{42} - \frac{6}{63} = \frac{17}{21} - \frac{2}{21} = \frac{15}{21} = \frac{5}{7}$$

Substitution of $x = -\frac{1}{7}$ into the right-hand side of the original equation we find

$$\frac{-\frac{1}{7}+3}{4} = \frac{\frac{-1+21}{7}}{4} = \frac{20}{28} = \frac{5}{7}$$

We see that, with $x = -\frac{1}{7}$ both sides are equal and so the solution is correct.

Example

Solve $\frac{4-5x}{6} - \frac{1-2x}{3} = \frac{13}{42}$.

Solution

First of all remember that the division line means divide all of $4-5x$ by 6. So let's put in brackets to be absolutely clear:

$$\frac{(4-5x)}{6} - \frac{(1-2x)}{3} = \frac{13}{42}$$

Now we need a common denominator for the denominators 6, 3 and 42. Note that both 6 and 3 will divide into 42 so choose 42 as the common denominator. Multiply everything by 42. So we have

$$\frac{42(4-5x)}{6} - \frac{42(1-2x)}{3} = 42 \times \frac{13}{42}$$

Simplifying each term

$$\frac{{}^{7}\cancel{42}(4-5x)}{\cancel{6}_1} - \frac{{}^{14}\cancel{42}(1-2x)}{\cancel{3}_1} = {}^{1}\cancel{42} \times \frac{13}{\cancel{42}_1}$$

Now multiply out the brackets and simplify the left-hand side.

$$28 - 35x - 14 + 28x = 13$$

From which

$$\begin{aligned} 14 - 7x &= 13 \\ -7x &= -1 \\ x &= \frac{-1}{-7} \\ &= \frac{1}{7} \end{aligned}$$

So, the solution is $x = \frac{1}{7}$. Again this should be checked.

Substitution of $x = \frac{1}{7}$ into the left-hand side of the original equation gives

$$\frac{4 - \frac{5}{7}}{6} - \frac{1 - \frac{2}{7}}{3}$$

Simplifying we find this equals

$$\frac{28-5}{42} - \frac{7-2}{21} = \frac{23}{42} - \frac{5}{21} = \frac{23}{42} - \frac{10}{42} = \frac{13}{42}$$

and since this is the same as the right-hand side of the original equation the solution is correct.

Exercises

3. Solve the equations

   a) $5 + \frac{x}{3} = 7$ b) $\frac{1}{2}x - 1 = 5$ c) $\frac{3}{4}x - 2 = \frac{1}{3}x + 3$

   d) $4 - \frac{2}{3}x = \frac{x-6}{5}$ e) $\frac{x+2}{3} = \frac{1-2x}{5}$ f) $\frac{5x+1}{2} - \frac{x-2}{6} = \frac{2x+4}{3}$

5. Another form of a linear equation in one variable

In this final section we have a look at some equations which at first sight appear not to be linear equations. However, with some algebraic manipulation they can be recast in a more familiar form.

Example

Solve $\frac{3}{5} = \frac{6}{x}$.

Solution

Again, we need a common denominator. We need a quantity that will be divisible by 5 and by x. The obvious choice is $5x$. So let's multiply both sides by $5x$ and simplify.

$$^{1}5x \times \frac{3}{5_1} = 5x^{1} \times \frac{6}{x_1}$$

And so

$$3x = 30$$

Finally x must be equal to 10.

We now look at another way of solving this equation:

$$\frac{3}{5} = \frac{6}{x}$$

If two fractions are equal, they are also equal if we invert them.

$$\frac{5}{3} = \frac{x}{6}$$

This makes it easier still because all we need to do now is multiply by the common denominator and we can see what the common denominator is. It's quite clearly 6. Multiply by the common denominator of 6 and simplify the result.

$$ {}^{2}6 \times \frac{5}{3_1} = {}^{1}6 \times \frac{x}{6_1} $$

so that

$$10 = x$$

from which $x = 10$ as before.

Example

Solve $\frac{5}{3x} = \frac{25}{27}$.

Solution

We will tackle this by inverting each fraction.

$$\frac{3x}{5} = \frac{27}{25}$$

There is now a common denominator of 25. Multiply both sides by 25 and simplify to get

$$\begin{aligned} 15x &= 27 \\ x &= \frac{27}{15} \\ &= \frac{9}{5} \end{aligned}$$

The answer can be left like this or written as the mixed number $1\frac{4}{5}$.

Now some of you may not like the idea of flipping over the fractions. So let's tackle this in another way. So again, to solve

$$\frac{5}{3x} = \frac{25}{27}$$

Let's look for a common denominator between $3x$ and 27. So we want something $3x$ will divide into exactly and something that 27 will divide into exactly. Such a quantity is $27x$. So that's going to be our common denominator. Multiply both sides by $27x$.

$$27x \times \frac{5}{3x} = 27x \times \frac{25}{27}$$

So

$$45 = 25x$$

from which

$$x = \frac{45}{25} = \frac{9}{5}$$

Example

Solve $\frac{19x}{7} = \frac{57}{49}$.

Solution

The common denominator of 7 and 49 is 49. Multiplying both sides by 49 and simplifying:

$$49 \times \frac{19x}{7} = 49 \times \frac{57}{49}$$

So

$$7 \times 19x = 57$$

and dividing each side by 19:

$$7x = 3$$

which means

$$x = \frac{3}{7}$$

The important thing in dealing with these kind of equations and any kind of equations is to remember that the equals sign represents a balance. What it tells you is that what's on the left-hand side is exactly equal to what's on the right-hand side. So whatever you do to one side you have to do to the other side and you must follow the rules of arithmetic when you do it.

Exercises

4. Solve the following equations.

a) $6x + 2 = 29 - 3x$ b) $\frac{1}{3}x + 4 = \frac{4x - 1}{5}$ c) $\frac{3x}{4} = \frac{2}{5}$

d) $\frac{8}{x} = 2$ e) $\frac{7}{3x} = 2$ f) $\frac{3}{x + 1} = \frac{6}{5x - 1}$

Answers

1. a) 4 b) 5 c) 3/5
   d) -2 e) 3 f) 2
   g) 2 h) -1 i) 4

2. a) 2 b) $-2/3$ c) -1

3. a) 6 b) 12 c) 12
   d) 6 e) $-7/11$ f) 3/10

4. a) 3 b) 9 c) 8/15
   d) 4 e) 7/6 f) 1

Simplifying fractions

mc-simpfrac-2009-1

The ability to simplify fractions and to write them in equivalent forms is an essential mathematical skill required of all engineers and physical scientists. This unit explains how these processes are carried out.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature. To help you to achieve this, the unit includes a number of such exercises.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- simplify algebraic expressions by cancelling common factors

Contents

1. Introduction

When we come across fractions one of the things we often have to do is to study them to see if we can put them in a simpler form. In fact, we look to see if we can put them in a form that might be called their lowest terms or simplest form.
For instance, suppose we have a fraction $\frac{12}{36}$. This is not in its lowest terms. 12 will divide into both the numerator, 12, and the denominator 36. So, we can divide both numerator and denominator by 12 to give

$$\frac{\cancel{12}^{\,1}}{\cancel{36}_{\,3}} = \frac{1}{3}$$

The two fractions $\frac{12}{36}$ and $\frac{1}{3}$ have the same value; they are equivalent fractions.
We want to carry out similar operations with algebraic expressions. Instead of looking for numbers which will divide into both the numerator and the denominator, we now look for algebraic expressions which will divide into both.

2. Some introductory examples

Example

Suppose we wish to write

$$\frac{3x^3}{x^5}$$

in its simplest form.

Clearly there are x terms in both the numerator and the denominator. Because x^5 can be considered as $x^3 \times x^2$ we observe a common factor of x^3 in both numerator and denominator.

$$\frac{3x^3}{x^5} = \frac{3x^3}{x^3x^2}$$

Now we can divide top and bottom by x^3 to remove this common factor.

$$\frac{3x^3}{x^5} = \frac{3}{x^2}$$

With practice you will be able to miss out the middle step and go straight from $\frac{3x^3}{x^5}$ to $\frac{3}{x^2}$

Example

Suppose we wish to write

$$\frac{x^3y^3}{x^2y}$$

in its simplest form.

We look for expressions which are common to both the top and the bottom. Because x^3 can be considered as $x \times x^2$ we observe that there is a common factor of x^2. Similarly because y^3 can be considered as $y \times y^2$ we observe that there is also a common factor of y.
We can divide top and bottom by x^2, and then by y to give

$$\frac{x^3y^3}{x^2y} = \frac{xy^2}{1} = xy^2$$

Example

Suppose we wish to write

$$\frac{-16x^2y^2}{4x^3y^2}$$

in its simplest form.

Because $-16 = -4 \times 4$ we see there is a common factor of 4 in both numerator and denominator. Note also that both x^2 and y^2 are common factors. We can divide top and bottom, in turn, by each of these common factors:

$$\frac{-16x^2y^2}{4x^3y^2} = \frac{-4}{x}$$

Exercise 1

Express each of the following fractions in its simplest form.

a) $\frac{12x^2}{x^5}$ b) $\frac{5x^3y^2}{10xy^4}$ c) $\frac{4x^8}{2x^6}$ d) $\frac{9y^2}{3y}$

e) $\frac{4x^2y^{10}}{12x^4y^5}$ f) $\frac{3(x+y)^5}{12(x+y)}$ g) $\frac{-9xy}{27x^2}$ h) $\frac{-14xy^5}{2x^3y^2}$

3. Examples requiring factorisation of the numerator or denominator

Example

Suppose we wish to write

$$\frac{x^2 - 2xy}{x}$$

in its simplest form. In this Example the common factor may not be obvious. We start by asking is there a common factor in the numerator. In both terms x^2 and $2xy$ there is a factor of x, so we can factorise the numerator.

$$\frac{x^2 - 2xy}{x} = \frac{x(x-2y)}{x}$$

In this form we can see that there is a common factor of x in both the numerator and the denominator. We can divide top and bottom by x to remove this:

$$\frac{x^2 - 2xy}{x} = \frac{x(x-2y)}{x} = \frac{x-2y}{1} = x - 2y$$

So $\frac{x^2 - 2xy}{x}$ is equivalent to $x - 2y$.

Example

Suppose we wish to write

$$\frac{x^6 - 7x^5 + 4x^8}{x^2}$$

in its simplest form.

As in the previous example we look for common factors in the numerator. Observe there is a common factor of x^5 and so we factorise the numerator as follows:

$$\frac{x^6 - 7x^5 + 4x^8}{x^2} = \frac{x^5(x - 7 + 4x^3)}{x^2}$$

There is clearly a common factor of x^2 in the numerator and the denominator, and so dividing top and bottom by this factor we find

$$\frac{x^6 - 7x^5 + 4x^8}{x^2} = \frac{x^5(x - 7 + 4x^3)}{x^2} = \frac{x^3(x - 7 + 4x^3)}{1} = x^3(x - 7 + 4x^3)$$

There is no need to remove the brackets in the answer.

Example

Suppose we wish to write

$$\frac{x + 1}{x^2 + 3x + 2}$$

in its simplest form.

There are no obvious common factors. But, the denominator is a quadratic and so we try to factorise it. It may turn out that one of the factors is the same as the term at the top. So factorising the denominator

$$\frac{x + 1}{x^2 + 3x + 2} = \frac{(x + 1)}{(x + 2)(x + 1)}$$

In this form we see a common factor of $(x + 1)$. Dividing top and bottom by this factor removes it:

$$\frac{x + 1}{x^2 + 3x + 2} = \frac{\cancel{(x + 1)}^{1}}{(x + 2)\cancel{(x + 1)}_{1}} = \frac{1}{x + 2}$$

Example

Suppose we wish to write

$$\frac{a^2 - 11a + 30}{a - 5}$$

in its simplest form. Again, there is a quadratic which we can try to factorise:

$$\frac{a^2 - 11a + 30}{a - 5} = \frac{(a - 6)(a - 5)}{(a - 5)}$$

Notice the common factor of $(a - 5)$. We can divide top and bottom by this factor to remove it.

$$\frac{a^2 - 11a + 30}{a - 5} = \frac{(a - 6)\cancel{(a - 5)}^{1}}{\cancel{(a - 5)}_{1}} = a - 6$$

4. A mistake to be avoided

Dividing top and bottom by common factors is often loosely referred to as *cancelling* common factors, because, as we have seen, we cancel them out during our working. A common mistake that students sometimes make is to cancel out terms which ought not to be cancelled. We must not be tempted to cancel numbers straightaway. We can only cancel common factors. For example, we have seen students take an expression such as

$$\frac{3x^2 + 10x + 3}{x + 3}$$

and simply cancel out all the 3's. This would certainly be incorrect. Let us look at a numerical example to see why.

Suppose we have $\frac{5 + 3}{3 + 1}$. Cancelling the 3's we would find

$$\frac{5 + \cancel{3}^{\,1}}{{}_1\cancel{3} + 1} = \frac{6}{2} = 3$$

However, the true value is obtained correctly as

$$\frac{5 + 3}{3 + 1} = \frac{8}{4} = \frac{\cancel{8}^{\,2}}{\cancel{4}_{\,1}} = 2$$

We see that simply cancelling 3's leads to an incorrect result. Never do this. You must only cancel common factors.

The correct way to handle $\frac{3x^2 + 10x + 3}{x + 3}$ is to factorise the numerator and then cancel any common factors:

$$\frac{3x^2 + 10x + 3}{x + 3} = \frac{(3x + 1)(x + 3)}{(x + 3)} = \frac{(3x + 1)\,\cancel{(x + 3)}^{\,1}}{\cancel{(x + 3)}_{\,1}} = 3x + 1$$

Example

Suppose we wish to write

$$\frac{6x^3 - 7x^2 - 5x}{2x + 1}$$

in its simplest form. Observe the common factor of x in each of the terms in the numerator. We proceed as follows.

$$\frac{6x^3 - 7x^2 - 5x}{2x + 1} = \frac{x(6x^2 - 7x - 5)}{2x + 1} = \frac{x(3x - 5)\,\cancel{(2x + 1)}^{\,1}}{\cancel{(2x + 1)}} = x(3x - 5)$$

Again, it is better to leave this answer in its factorised form.

Example

Suppose we wish to write

$$\frac{x^3 - 1}{x - 1}$$

in its simplest form. It is not obvious how to proceed here.

However the numerator will factorise to

$$x^3 - 1 = (x-1)(x^2+x+1)$$

You should remove the brackets yourself to check this. Then,

$$\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{(x-1)} = x^2+x+1$$

Exercise 2

Express each of the following fractions in its simplest form.

a) $\dfrac{2x+4}{12}$ b) $\dfrac{xy-3x}{x^3}$ c) $\dfrac{x^2y+y^2x}{x^2y^2}$ d) $\dfrac{x^3-2x^2+5x}{x^4}$

e) $\dfrac{5x^2+25x}{100x^3}$ f) $\dfrac{x+2}{x^2-x-6}$ g) $\dfrac{x^2+5x}{x+5}$ h) $\dfrac{x+2y}{x^2+3xy+2y^2}$

Answers

Exercise 1

a) $\dfrac{12}{x^3}$ b) $\dfrac{x^2}{2y^2}$ c) $2x^2$ d) $3y$

e) $\dfrac{y^5}{3x^2}$ f) $\dfrac{(x+y)^4}{4}$ g) $\dfrac{-y}{3x}$ h) $\dfrac{-7y^3}{x^2}$

Exercise 2

a) $\dfrac{x+2}{6}$ b) $\dfrac{y-3}{x^2}$ c) $\dfrac{x+y}{xy}$ d) $\dfrac{x^2-2x+5}{x^3}$

e) $\dfrac{x+5}{20x^2}$ f) $\dfrac{1}{x-3}$ g) x h) $\dfrac{1}{x+y}$

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Substitution & Formulae

mc-subsandformulae-2009-1

In mathematics, engineering and science, formulae are used to relate physical quantities to each other. They provide rules so that if we know the values of certain quantities, we can calculate the values of others. In this unit we discuss several formulae and illustrate how they are used.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- substitute numbers into formulae in order to calculate various physical quantities

Contents

1. Introduction

A formula is a recipe, or a rule, for doing something. In mathematics, a formula gives a relationship between different quantities. When there is more than one, we use the word formulae. There are many standard formulae which have been derived by mathematicians and scientists. In this unit we will see lots of formulae and practice using them.

2. The formula for the area of a circle

The formula which tells us the area of a circle is $A = \pi r^2$. Here, r is the radius of the circle, and A is its area. To calculate the area of a circle we square its radius and multiply the result by π. (In the examples which follow we shall approximate π by 3.142.)

Key Point

the area, A, of a circle with radius r:

$$A = \pi r^2$$

Example

Suppose we wish to find the area of a circular lawn, of radius 3m, in order that we can turf it. We can calculate the area by substituting the value $r = 3$ in the formula:

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \times 3^2 \\ &= \pi \times 9 \end{aligned}$$

Taking π as 3.142 we can use a calculator to obtain $A = 28.3$ correct to 1 decimal place. So we would need to purchase approximately 29 m^2 of turf in order to cover the lawn.

3. The formula for the volume of a sphere

The formula for the volume, V, of a sphere, or ball, having radius r is $V = \frac{4}{3}\pi r^3$.

Key Point

the volume, V, of a sphere of radius r:

$$V = \frac{4}{3}\pi r^3$$

Example

Suppose we wish to find the volume of a football which has radius 10cm. Then

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi \times 10^3 \\ &= 4189 \quad \text{(to 4 significant figures)} \end{aligned}$$

So the volume of the football is 4189 cm^3.

Exercises 1

Given below are a selection of formulae relating to areas, perimeters and volumes of geometric shapes.

Circles: $A = \pi r^2 = \pi \frac{d^2}{4}$ and $C = 2\pi r = \pi d$

where A is the area, C is the circumference, r is the radius and d is the diameter.

Rectangles: $A = wh$ and $P = 2(w + h)$

where A is the area, P is the perimeter, w is the width and h is the height.

Spheres: $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$

where V is the volume, A is the surface area and r is the radius.

Cylinders: $V = \pi r^2 h$, $C = 2\pi rh$ and $A = 2\pi r(r + h)$

where V is the volume, C is the curved surface area, A is the surface area of a solid cylinder (including the ends), r is the radius and h is the height.

In the following exercises give your answer correct to 1 decimal place.

1. What is the area of a circle with radius 3cm?
2. What is the circumference of a circle with diameter 5cm?
3. What is the perimeter of a rectangle with width 5cm and height 8 cm?
4. What is the volume of a sphere with radius 2cm?
5. What is the curved surface area of a cylinder with radius 10cm and height 40cm?
6. What is the area of a rectangle with width 3.5 cm and height 5.2 cm?
7. What is the surface area of a sphere with radius 4.7 cm?
8. What is the total surface area of a cylindrical can with radius 5.3 cm and height 8.1 cm?
9. What is the total surface area of a solid hemisphere with radius 7.3 cm?
10. A strong man's barbell is made up of two spheres, each with radius 15cm joined by a cylinder of length 85cm and radius 2cm. What is the volume of the barbell?

4. Newton's second law of motion

This law relates the quantities force F, mass m, and acceleration a, and states $F = ma$.

Key Point

Newton's second law

$$F = ma$$

where F= force, m=mass, a=acceleration.

Example

We wish to calculate the force on a circus performer who is fired from a cannon.

His mass is 60 kg and his acceleration on leaving the cannon is 2.5 ms^{-2}.

Using Newton's second law:

$$\begin{aligned} F &= ma \\ &= 60 \times 2.5 \\ &= 150 \end{aligned}$$

So the force on the performer is 150 newtons. (The newton is the unit of force, when the mass is measured in kilograms and the acceleration is measured in metres per second per second).

5. The equations of motion

One of the so-called 'equations of motion' is $v = u + at$ where v = final speed, u = initial speed, a = acceleration, and t = time.
Example
Suppose an object has an initial speed 5ms^{-1} and moves with an acceleration of 2ms^{-2} for 3 seconds. We want to calculate its final speed.

$$\begin{aligned} v &= u + at \\ &= 5 + 2 \times 3 \\ &= 11 \end{aligned}$$

So its final speed is 11ms^{-1}.

Here it is worth pointing out the correct order of carrying out the operations in the calculation of $5 + 2 \times 3$. The multiplication is carried out before the addition. You may recall the BODMAS rule which states the order in which operations should be carried out.

Brackets
pOwers
Division
Multiplication
Addition
Subtraction

and so multiplication is carried out before addition.

Another equation of motion states $v^2 = u^2 + 2as$. As before, v = final speed, u = initial speed, a = acceleration, s = distance travelled.

Example

Suppose a stone is dropped from a cliff which is 100m high. Because the stone is simply dropped, rather than thrown, its initial speed is zero, and so $u = 0$. Its acceleration will be the acceleration due to gravity which is $a = 9.8$ms^{-2}. Suppose we wish to calculate the final speed just before the stone hits the water at the foot of the cliff. The stone travels a distance of 100m before hitting the water and so $s = 100$. We must substitute all these known values into the formula.

$$\begin{aligned} v^2 &= u^2 + 2as \\ &= 0^2 + 2 \times 9.8 \times 100 \\ &= 1960 \end{aligned}$$

So to calculate v, we need to find the square root of this result: $v = \sqrt{1960} = 44$ ms^{-1} (to two significant figures).

A third equation of motion is $s = ut + \frac{1}{2}at^2$ where s = distance travelled, t= time, u = initial speed, a = acceleration.

Example

We wish to determine the depth of a well by dropping a stone down it.

We find that it takes 3 seconds for the stone to reach the bottom of the well. As in the previous example, the initial speed is zero and the acceleration due to gravity is 9.8ms^{-2}.

We substitute the given values into the formula.

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ &= 0 \times 3 + \frac{1}{2} \times 9.8 \times 3^2 \\ &= 44 \end{aligned}$$

So the depth of the well is 44m (to the nearest whole number).

Key Point

The equations of motion with constant acceleration:

$$v = u + at \qquad v^2 = u^2 + 2as \qquad s = ut + \frac{1}{2}at^2$$

where u = initial speed, v = final speed, a = acceleration, t = time, s = distance travelled.

6. The formula for kinetic energy

If an object has mass m and moves with speed v, its kinetic energy (K.E.) is given by the formula $\text{K.E.} = \frac{1}{2}mv^2$.

Key Point

$$\text{Kinetic Energy} = \frac{1}{2}mv^2$$

where m = mass, and v = speed.

Example

A sprinter of mass 70 kg runs at a speed of 10ms^{-1}.
A truck has mass 2000 kg and moves with a speed of 20ms^{-1}.

Let us compare their kinetic energies:

For the sprinter: $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 70 \times 10^2 = 3500$

For the truck: $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 2000 \times 20^2 = 400\,000$

So the kinetic energy of the truck is more than 100 times greater than that of the sprinter.

When m is measured in kilograms, and v is measured in metres per second, the unit of kinetic energy is the joule.

7. The period of a pendulum

The formula which tells us the time for a complete swing of a pendulum, i.e. its period, T, is $T = 2\pi\sqrt{\frac{\ell}{g}}$. Here ℓ is the length of the pendulum, and g is the acceleration due to gravity which is 9.8 ms^{-2}.

Key Point

The period T of a pendulum $= 2\pi\sqrt{\frac{\ell}{g}}$

where ℓ = length of pendulum, and g = acceleration due to gravity.

Example

Suppose we wish to find the period of a grandfather clock which has a pendulum of length 1m.

$$
\begin{aligned}
T &= 2\pi\sqrt{\frac{\ell}{g}} \\
&= 2\pi\sqrt{\frac{1}{9.8}} \\
&= 2\pi \times 0.319
\end{aligned}
$$

Approximating π by 3.142 we find $T = 2.007$. So the period of the pendulum is 2 seconds, to the nearest second.

Exercises 2

Some formulae from mechanics are given below.

Constant Acceleration

$v = u + at$, $v^2 = u^2 + 2as$ and $s = ut + \frac{1}{2}at^2$

where u is the initial speed, v is the final speed, t is time, s is distance travelled and a is acceleration. In the SI system of units u and v are measured in metres/second (m/s or ms^{-1}), t in seconds (s), s in metres (m) and a in metres per second squared (m/s^2 or ms^{-2}).

Work and Energy

$W = Fd$, $K.E. = \frac{1}{2}mv^2$, $P.E. = mgh$

where W is work done, $K.E.$ is kinetic energy, $P.E.$ is potential energy, F is force, d is distance moved by the force, m is mass, v is speed, g is acceleration due to gravity and h is height. In the SI system of units W, $K.E.$ and $P.E.$ are measured in Joules (J), F in Newtons (N), d in metres (m), m in kilograms (kg), v in metres per second (m/s or ms^{-1}), g in metres per second squared (m/s^2 or ms^{-2}) and h in metres (m).

Throughout these exercises you should take the acceleration due to gravity, g, as 9.8 m/s^2.

Give your answers correct to 2 decimal places.

1. A car, starting at rest, accelerates at 5 m/s^2 for 3 seconds. What is its final speed (in m/s)?
2. How far does the car in the previous question travel?
3. A stone is dropped from the top of a building 32 m tall. Its acceleration is that due to gravity. What is its speed (in m/s) when it hits the ground?
4. If the mass of the stone in the previous question is 0.1 kg, what is its potential energy when it is released?
5. What is the kinetic energy of the stone from the previous two questions when it hits the ground?
6. What is the work done by a horizontal force of 10N in moving an object horizontally through 5m.
7. Determine the kinetic energy of a mass of 6.2 kg moving at a speed of 3.1 m/s.

8. Formulae used to generate terms of a sequence

There are two simple ways of generating sequences of numbers.

(a) by a rule which tells you how to calculate the next term from the previous term (or possibly from the previous two or more terms)

(b) by a formula which tells you how to calculate each term independently.

Consider the sequence

$$2, 4, 6, 8, \ldots$$

This is generated by the rule 'add 2 to the previous term to get the next term, given that the first term is 2'. Alternatively it can be generated from the formula $u_n = 2n$ (where u_n means the nth term of the sequence). So, for example, to find the 5th term we put $n = 5$ to get $u_5 = 2 \times 5 = 10$.

When using a rule to generate a sequence, if you want to work out a specific term in the sequence you must work out all the previous terms. But when you have a formula you can work out any term you like without having to calculate others.

A famous sequence is the Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, \ldots$.

Here the rule is 'add the previous two terms together to get the next term, given that the first two terms are 0 and 1'. There is a formula which can be used to generate each term independently but it requires some quite complex mathematics to determine it.

The accompanying video shows how a formula can be used in an impressive magic trick.

Exercises 3

In the exercises below, give answers which are not whole numbers correct to 2 decimal places. Note that questions 7 to 10 involve the exponential, natural logarithm and sine functions. If you are not familiar with these functions then simply omit these questions.

1. Calculate the 5th term of the sequence generated by the formula $u_n = 3n + 1$.
2. Calculate the 10th term of the sequence generated by the formula $u_n = 54 - 2n$.
3. Calculate the 12th term of the sequence generated by the formula $u_n = n^2 - 10n + 1$.
4. Calculate the 5th term of the sequence generated by the formula $u_n = \frac{1}{2}n(n + 1)$.
5. Calculate the 8th term of the sequence generated by the formula $u_n = \frac{1}{6}n(n + 1)(2n + 1)$.
6. Calculate the 6th term of the sequence generated by the formula $u_n = 2\sqrt{n + 3}$ (take the positive square root).
7. Calculate the 4th term of the sequence generated by the formula $u_n = e^{-n/4}$.
8. Calculate the 9th term of the sequence generated by the formula $u_n = \frac{n^2}{10}$.
9. Calculate the 3rd term of the sequence generated by the formula $u_n = \sqrt{\frac{\ln n^3}{n^2 - 1}}$
10. Calculate the 11th term of the sequence generated by the formula $u_n = \sin\left(\frac{n\pi}{3}\right)$, where the angle is measured in radians.

Answers

Exercise 1

1)	28.3 cm^2	2)	19.6 cm	3)	26 cm	4)	33.5 cm^3
5)	2513.3 cm^2	6)	18.2 cm^2	7)	277.6 cm^2	8)	446.2 cm^2
9)	502.2 cm^2	10)	29342.5 cm^3				

Exercise 2

1)	15 m/s	2)	22.5 m	3)	25.04 m/s	4)	31.36 J
5)	31.36 J	6)	50 J	7)	29.79 J		

Exercise 3

1)	16	2)	34	3)	25	4)	15
5)	204	6)	6	7)	0.37	8)	2.09
9)	0.64	10)	-0.87				

Simultaneous linear equations

mc-simultaneous-2009-1

The purpose of this section is to look at the solution of simultaneous linear equations. We will see that solving a pair of simultaneous equations is equivalent to finding the location of the point of intersection of two straight lines.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that all this becomes second nature. To help you to achieve this, the unit includes a number of such exercises.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- solve pairs of simultaneous linear equations
- recognise that this is equivalent to finding the point of intersection of two straight line graphs

Contents

1. Introduction

The purpose of this section is to look at the solution of elementary simultaneous linear equations.

Before we do that, let's just have a look at a relatively straightforward single equation. The equation we are going to look at is

$$2x - y = 3$$

This is a linear equation. It is a linear equation because there are no terms involving x^2, y^2 or $x \times y$, or indeed any higher powers of x and y. The only terms we have got are terms in x, terms in y and some numbers. So this is a linear equation.

We can rearrange it so that we obtain y on its own on the left hand side. We can add y to each side so that we get

$$2x = 3 + y$$

Now let's take 3 away from each side.

$$2x - 3 = y$$

This gives us an expression for y: namely $y = 2x - 3$.

Suppose we choose a value for x, say $x = 1$, then y will be equal to:

$$y = 2 \times 1 - 3 = -1$$

Suppose we choose a different value for x, say $x = 2$.

$$y = 2 \times 2 - 3 = 1$$

Supppose we choose another value for x, say $x = 0$.

$$y = 2 \times 0 - 3 = -3$$

For every value of x we can generate a value of y.

We can plot these as points on a graph. We can plot the first as the point $(1, -1)$. We can plot the second one as the point $(2, 1)$, and the third one as the point $(0, -3)$ and so on. Plotting the points on a graph, as shown in Figure 1, we see that these three points lie on a straight line. This is the line with equation $y = 2x - 3$. It is a straight line and this is another reason for calling the equation a linear equation.

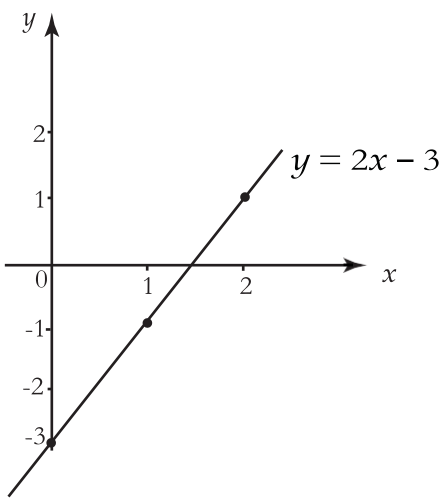


Figure 1. Graph of $y = 2x - 3$.

Suppose we take a second linear equation $3x + 2y = 8$ and plot its graph on the same figure. A quick way to achieve this is as follows.

When $x = 0$, $2y = 8$, so $y = 4$. Therefore the point $(0, 4)$ lies on the line.

When $y = 0$, $3x = 8$, so $x = \frac{8}{3} = 2\frac{2}{3}$. Therefore the point $(2\frac{2}{3}, 0)$ lies on the line.

Because this is a linear equation we know its graph is a straight line, so we can obtain this by joining up the points. Both straightline graphs are shown in Figure 2.

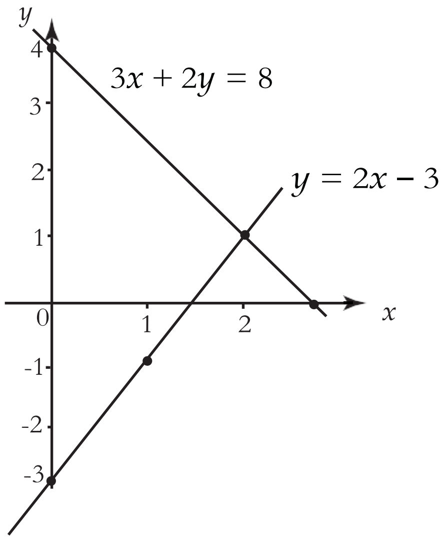


Figure 2. Graphs of $y = 2x - 3$ and $3x + 2y = 8$

When we solve a pair of simultaneous equations what we are actually looking for is the intersection of two straight lines because it is this point that satisfies both equations at the same time. From Figure 2 we see that this occurs at the point where $x = 2$ and $y = 1$.

Of course it could happen that we have two parallel lines; they would never meet, and hence the simultaneous equations would not have a solution. We shall observe this behaviour in one of the examples which follows.

Key Point

When solving a pair of simultaneous linear equations we are, in fact, finding a common point - the point of intersection of the two lines.

2. Solving simultaneous equations - method of substitution

How can we handle the two equations algebraically so that we do not have to draw graphs? We are going to look at two methods of solution. In this Section we will look at the first method - the method of substitution.

Let us return to the two equations we met in Section 1.

$$2x - y = 3 \qquad (1)$$

$$3x + 2y = 8 \qquad (2)$$

By rearranging Equation (1) we find

$$y = 2x - 3 \qquad (3)$$

We can now substitute this expression for y into Equation (2).

$$\begin{aligned} 3x + 2(2x - 3) &= 8 \\ 3x + 4x - 6 &= 8 \\ 7x - 6 &= 8 \\ 7x &= 14 \\ x &= 2 \end{aligned}$$

Finally, using Equation (3), $y = 2 \times 2 - 3 = 1$. So $x = 2$, $y = 1$ is the solution to the pair of simultaneous equations.

This solution should always be checked by substituting back into both original equations to ensure that the left- and right- hand sides are equal for these values of x and y. So, with $x = 2, y = 1$, the left-hand side of Equation (1) is $2(2) - 1 = 3$, which is the same as the right-hand side. With $x = 2$, $y = 1$, the left-hand side of Equation (2) is $3(2) + 2(1) = 8$, which is the same as the right-hand side.

Example

Let's have a look at another example using this particular method.

The example we are going to use is

$$7x + 2y = 47 \qquad (1)$$

$$5x - 4y = 1 \qquad (2)$$

Now we need to make a choice. We need to choose one of these two equations and re-arrange it to obtain an expression for y, or if we wish, for x

The choice is entirely ours and we have to make the choice based upon what we feel will be the simplest. Looking at a pair of equations like this, it is often difficult to know which is the simplest.

Let's choose Equation (2) and rearrange it to find an expression for x.

$$\begin{aligned} 5x - 4y &= 1 \\ 5x &= 1 + 4y && \text{by adding } 4y \text{ to each side} \\ x &= \frac{1 + 4y}{5} && \text{by dividing both sides by 5} \end{aligned}$$

We now use this expression for x and substitute it in Equation (1).

$$7\;\frac{1 + 4y}{5} + 2y = 47$$

Now multiply throughout by 5. Why? Because we want to get rid of the fraction and the way to do that is to multiply everything by 5.

$$7(1 + 4y) + 10y = 235$$

Now we need to multiply out the brackets

$$7 + 28y + 10y = 235$$

Gather the y's and subtract 7 from each side to get

$$38y = 228$$

So

$$y = \frac{228}{38} = 6$$

So we have established that $y = 6$. Having done this we can substitute it back into the equation that we first had for x.

$$x = \frac{1 + 4y}{5} = \frac{1 + 24}{5}$$

and so

$$x = 5$$

So again, we have our pair of values - our solution to the pair of simultaneous equations. In order to check that our solution is correct these values should be substituted into both equations to ensure they balance. So, with $x = 5$, $y = 6$, the left-hand side of Equation (1) is $7(5)+2(6) = 47$, which is the same as the right-hand side. With $x = 5$, $y = 6$, the left-hand side of Equation (2) is $5(5) - 4(6) = 1$, which is the same as the right-hand side.

Exercises

1. Solve the following pairs of simultaneous equations:

a) $\begin{aligned} y &= 2x + 3 \\ y &= 5x - 3 \end{aligned}$ b) $\begin{aligned} y &= 3x - 1 \\ 2x + 4y &= 10 \end{aligned}$ c) $\begin{aligned} 6x + y &= 4 \\ 5x + 2y &= 1 \end{aligned}$

d) $\begin{aligned} x - 3y &= 1 \\ 2x + 5y &= 35 \end{aligned}$ e) $\begin{aligned} 2x + \tfrac{1}{3}y &= 1 \\ 3x + 5y &= 6 \end{aligned}$ f) $\begin{aligned} 4x + 3y &= 5 \\ 2x - \tfrac{3}{4}y &= 1 \end{aligned}$

3. Solving simultaneous equations - method of elimination

We illustrate the second method by solving the simultaneous linear equations:

$$7x + 2y = 47 \quad (1)$$

$$5x - 4y = 1 \quad (2)$$

We are going to multiply Equation (1) by 2 because this will make the magnitude of the coefficients of y the same in both equations. Equation (1) becomes

$$14x + 4y = 94 \quad (3)$$

If we now add Equation (2) and Equation (3) we will find that the terms involving y disappear:

$$\begin{array}{rrrrcl} & 5x & - & 4y & = & 1 \\ + & 14x & + & 4y & = & 94 \\ \hline & 19x & & & = & 95 \end{array}$$

and so

$$x = \frac{95}{19} = 5$$

Now that we have a value for x we can substitute this into Equation (2) in order to find y. Substituting

$$\begin{aligned} 5x - 4y &= 1 \\ 5 \times 5 - 4y &= 1 \\ 25 &= 4y + 1 \\ 24 &= 4y \\ y &= 6 \end{aligned}$$

The solution is $x = 5$, $y = 6$.

4. Examples

Solve the simultaneous equations

$$3x + 7y = 27 \quad (1)$$

$$5x + 2y = 16 \quad (2)$$

We will multiply Equation (1) by 5 and Equation (2) by 3 because this will make the coefficients of x in both equations the same.

$$15x + 35y = 135 \quad (3)$$

$$15x + 6y = 48 \quad (4)$$

If we now subtract Equation (4) from Equation (3) we can eliminate the terms involving x.

$$\begin{array}{rcrcl} & 15x & + & 35y & = & 135 \\ - & 15x & + & 6y & = & 48 \\ \hline & & & 29y & = & 87 \end{array}$$

from which

$$y = \frac{87}{29} = 3$$

If we substitute this result in Equation (1) we can find x.

$$\begin{aligned} 3x + 7y &= 27 \\ 3x + 21 &= 27 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

As before, the solution should be checked by substitution into the original equations. So, with $x = 2$, $y = 3$, the left-hand side of Equation (1) is $3(2) + 7(3) = 27$, which is the same as the right-hand side. With $x = 2$, $y = 3$, the left-hand side of Equation (2) is $5(2) + 2(3) = 16$, which is the same as the right-hand side.

All the examples that we have looked at so far have all had whole number coefficients; let's have a look at a couple that don't look like the ones we have just done.

Example

Solve the simultaneous equations

$$\begin{aligned} x &= 3y \\ \frac{x}{3} - y &= 34 \end{aligned}$$

First of all let us rearrange the first equation so that x and y terms are on the left. We will also multiply the second equation by 3 to remove the fraction. These operations give

$$\begin{aligned} x - 3y &= 0 \\ x - 3y &= 102 \end{aligned}$$

Notice that the terms on the left in both equations are exactly the same. If we subtract the equations we will find $0 = -102$. This does not make sense. Remember right at the beginning of this unit we explained that if two lines are parallel they will not intersect. This is the case here. There are no solutions.

Example

$$\frac{x}{5} - \frac{y}{4} = 0 \qquad (1)$$

$$3x + \frac{1}{2}y = 17 \qquad (2)$$

Observe that if both sides of Equation (1) are multiplied by 20 we can remove the fractions:

$$4x - 5y = 0 \qquad (3)$$

If Equation (2) is multiplied by 2 we can remove the fraction there too.

$$6x + y = 34 \qquad (4)$$

Now multiply Equation (4) by 5:

$$30x + 5y = 170 \qquad (5)$$

We can now add (3) and (5) to obtain

$$34x = 170$$
$$x = \frac{170}{34} = 5$$

Substituting this value into Equation (1) gives

$$\frac{x}{5} - \frac{y}{4} = 0$$
$$1 - \frac{y}{4} = 0$$

from which $y = 4$.

So the solution is: $x = 5, y = 4$. As before, this should be checked by substitution into the original equations. So, with $x = 5$, $y = 4$, the left-hand side of Equation (1) is $\frac{5}{5} - \frac{4}{4} = 0$, which is the same as the right-hand side. With $x = 5$, $y = 4$, the left-hand side of Equation (2) is $3(5) + \frac{1}{2}(4) = 17$, which is the same as the right-hand side.

<u>To summarise:</u>
A pair of simultaneous equations represent two straight lines. In effect when we solve them we are looking for the point where the two straight lines intersect. The method of elimination is much better to use than the first method.

Remember the answer you get can always be checked by substituting the pair of values into the original equations.

Exercises

2. Use elimination to solve the following pairs of simultaneous equations.

   a) $5x + 3y = 9$, $2x - 3y = 12$ b) $2x - 3y = 9$, $2x + y = 13$ c) $x + 7y = 10$, $3x - 2y = 7$

   d) $5x + y = 10$, $7x - 3y = 14$ e) $\frac{1}{3}x + y = \frac{10}{3}$, $2x + \frac{1}{4}y = \frac{11}{4}$ f) $3x - 2y = \frac{5}{2}$, $\frac{1}{3}x + 3y = -\frac{4}{3}$

3. Solve the following pairs of simultaneous equations by a method of your choice.

   a) $x = 3y$, $4x - 5y = 35$ b) $x = \frac{1}{3}y$, $2y - 6x = 9$ c) $7x + 3y = -15$, $12y - 5x = 39$

Answers

1. a) $x = 2, y = 7$ b) $x = 1, y = 2$ c) $x = 1, y = -2$
   d) $x = 10, y = 3$ e) $x = 1/3, y = 1$ f) $x = 3/4, y = 2/3$

2. a) $x = 3, y = -2$ b) $x = 6, y = 1$ c) $x = 3, y = 1$
   d) $x = 2, y = 0$ e) $x = 1, y = 3$ f) $x = 1/2, y = -1/2$

3. a) $x = 15, y = 5$ b) no solution c) $x = -3, y = 2$

Quadratic Equations

mc-TY-quadeqns-1

This unit is about the solution of quadratic equations. These take the form $ax^2 + bx + c = 0$. We will look at four methods: solution by factorisation, solution by completing the square, solution using a formula, and solution using graphs

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- solve quadratic equations by factorisation
- solve quadratic equations by completing the square
- solve quadratic equations using a formula
- solve quadratic equations by drawing graphs

Contents

1. Introduction

This unit is about how to solve quadratic equations. A quadratic equation is one which must contain a term involving x^2, e.g. $3x^2$, $-5x^2$ or just x^2 on its own. It may also contain terms involving x, e.g. $5x$ or $-7x$, or $0.5x$. It can also have constant terms - these are just numbers: 6, -7, $\frac{1}{2}$.

It cannot have terms involving higher powers of x, like x^3. It cannot have terms like $\frac{1}{x}$ in it.

In general a quadratic equation will take the form

$$ax^2 + bx + c = 0$$

a can be any number excluding zero. b and c can be any numbers including zero. If b or c is zero then these terms will not appear.

Key Point

A quadratic equation takes the form

$$ax^2 + bx + c = 0$$

where a, b and c are numbers. The number a cannot be zero.

In this unit we will look at how to solve quadratic equations using four methods:

- solution by factorisation
- solution by completing the square
- solution using a formula
- solution using graphs

Factorisation and use of the formula are particularly important.

2. Solving quadratic equations by factorisation

In this section we will assume that you already know how to factorise a quadratic expression. If this is not the case you can study other material in this series where factorisation is explained.

Example

Suppose we wish to solve $3x^2 = 27$.

We begin by writing this in the standard form of a quadratic equation by subtracting 27 from each side to give $3x^2 - 27 = 0$.

We now look for common factors. By observation there is a common factor of 3 in both terms. This factor is extracted and written outside a pair of brackets. The contents of the brackets are adjusted accordingly:

$$3x^2 - 27 = 3(x^2 - 9) = 0$$

Notice here the difference of two squares which can be factorised as

$$3(x^2 - 9) = 3(x - 3)(x + 3) = 0$$

If two quantities are multiplied together and the result is zero then either or both of the quantities must be zero. So either

$$x - 3 = 0 \qquad \text{or} \qquad x + 3 = 0$$

so that

$$x = 3 \quad \text{or} \quad x = -3$$

These are the two solutions of the equation.

Example

Suppose we wish to solve $5x^2 + 3x = 0$.

We look to see if we can spot any common factors. There is a common factor of x in both terms. This is extracted and written in front of a pair of brackets:

$$x(5x + 3) = 0$$

Then either $x = 0$ or $5x + 3 = 0$ from which $x = -\frac{3}{5}$. These are the two solutions.

In this example there is no constant term. A common error that students make is to cancel the common factor of x in the original equation:

$$5x^{\not 2} + 3\not x = 0 \qquad \text{so that} \qquad 5x + 3 = 0 \qquad \text{giving } x = -\frac{3}{5}$$

But if we do this we lose the solution $x = 0$. In general, when solving quadratic equations we are looking for two solutions.

Example

Suppose we wish to solve $x^2 - 5x + 6 = 0$.

We factorise the quadratic by looking for two numbers which multiply together to give 6, and add to give -5. Now

$$-3 \times -2 = 6 \qquad -3 + -2 = -5$$

so the two numbers are -3 and -2. We use these two numbers to write $-5x$ as $-3x - 2x$ and proceed to factorise as follows:

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ x^2 - 3x - 2x + 6 &= 0 \\ x(x - 3) - 2(x - 3) &= 0 \\ (x - 3)(x - 2) &= 0 \end{aligned}$$

from which

$$x - 3 = 0 \qquad \text{or} \qquad x - 2 = 0$$

so that

$$x = 3 \qquad \text{or} \qquad x = 2$$

These are the two solutions.

Example

Suppose we wish to solve the equation $2x^2 + 3x - 2 = 0$.

To factorise this we seek two numbers which multiply to give -4 (the coefficient of x^2 multiplied by the constant term) and which add together to give 3.

$$4 \times -1 = -4 \qquad 4 + -1 = 3$$

so the two numbers are 4 and -1. We use these two numbers to write $3x$ as $4x - x$ and then factorise as follows:

$$\begin{aligned} 2x^2 + 3x - 2 &= 0 \\ 2x^2 + 4x - x - 2 &= 0 \\ 2x(x+2) - (x+2) &= 0 \\ (x+2)(2x-1) &= 0 \end{aligned}$$

from which

$$x + 2 = 0 \qquad \text{or} \qquad 2x - 1 = 0$$

so that

$$x = -2 \qquad \text{or} \qquad x = \frac{1}{2}$$

These are the two solutions.

Example

Suppose we wish to solve $4x^2 + 9 = 12x$.

First of all we write this in the standard form:

$$4x^2 - 12x + 9 = 0$$

We should look to see if there is a common factor - but there is not. To factorise we seek two numbers which multiply to give 36 (the coefficient of x^2 multiplied by the constant term) and add to give -12. Now, by inspection,

$$-6 \times -6 = 36 \qquad -6 + -6 = -12$$

so the two numbers are -6 and -6. We use these two numbers to write $-12x$ as $-6x - 6x$ and proceed to factorise as follows:

$$\begin{aligned} 4x^2 - 12x + 9 &= 0 \\ 4x^2 - 6x - 6x + 9 &= 0 \\ 2x(2x-3) - 3(2x-3) &= 0 \\ (2x-3)(2x-3) &= 0 \end{aligned}$$

from which

$$2x - 3 = 0 \qquad \text{or} \qquad 2x - 3 = 0$$

so that

$$x = \frac{3}{2} \qquad \text{or} \qquad x = \frac{3}{2}$$

These are the two solutions, but we have obtained the same answer twice. So we can have quadratic equations for which the solution is repeated.

Example

Suppose we wish to solve $x^2 - 3x - 2 = 0$.

We are looking for two numbers which multiply to give -2 and add together to give -3. Never mind how hard you try you will not find any such two numbers. So this equation will not factorise. We need another approach. This is the topic of the next section.

Exercise 1

Use factorisation to solve the following quadratic equations

a) $x^2 - 3x + 2 = 0$ b) $5x^2 = 20$ c) $x^2 - 5 = 4x$ d) $2x^2 = 10x$

e) $x^2 + 19x + 60 = 0$ f) $2x^2 + x - 6 = 0$ g) $2x^2 - x - 6 = 0$ h) $4x^2 = 11x - 6$

3. Solving quadratic equations by completing the square

Example

Suppose we wish to solve $x^2 - 3x - 2 = 0$.

In order to complete the square we look at the first two terms, and try to write them in the form $(\quad)^2$. Clearly we need an x in the brackets:

$(x + ?)^2$ because when the term in brackets is squared this will give the term x^2

We also need the number $-\frac{3}{2}$ which is half of the coefficient of x in the quadratic equation,

$\left(x - \frac{3}{2}\right)^2$ because when the term in brackets is squared this will give the term $-3x$

However, removing the brackets from $\left(x - \frac{3}{2}\right)^2$ we see there is also a term $\left(-\frac{3}{2}\right)^2$ which we do not want, and so we subtract this again. So the quadratic equation can be written

$$x^2 - 3x - 2 = \left(x - \frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2 - 2 = 0$$

Simplifying

$$\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 2 = 0$$

$$\left(x - \frac{3}{2}\right)^2 - \frac{17}{4} = 0$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{17}{4}$$

$$x - \frac{3}{2} = \frac{\sqrt{17}}{2} \quad \text{or} \quad -\frac{\sqrt{17}}{2}$$

$$x = \frac{3}{2} + \frac{\sqrt{17}}{2} \quad \text{or} \quad x = \frac{3}{2} - \frac{\sqrt{17}}{2}$$

We can write these solutions as

$$x = \frac{3 + \sqrt{17}}{2} \quad \text{or} \quad \frac{3 - \sqrt{17}}{2}$$

Again we have two answers. These are exact answers. Approximate values can be obtained using a calculator.

Exercise 2

a) Show that $x^2 + 2x = (x + 1)^2 - 1$.
Hence, use completing the square to solve $x^2 + 2x - 3 = 0$.

b) Show that $x^2 - 6x = (x - 3)^2 - 9$.
Hence use completing the square to solve $x^2 - 6x = 5$.

c) Use completing the square to solve $x^2 - 5x + 1 = 0$.

d) Use completing the square to solve $x^2 + 8x + 4 = 0$.

4. Solving quadratic equations using a formula

Consider the general quadratic equation $ax^2 + bx + c = 0$.

There is a formula for solving this: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. It is so important that you should learn it.

Key Point

Formula for solving $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We will illustrate the use of this formula in the following example.

Example

Suppose we wish to solve $x^2 - 3x - 2 = 0$.

Comparing this with the general form $ax^2 + bx + c = 0$ we see that $a = 1$, $b = -3$ and $c = -2$. These values are substituted into the formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times (-2)}}{2 \times 1} \\ &= \frac{3 \pm \sqrt{9 + 8}}{2} \\ &= \frac{3 \pm \sqrt{17}}{2} \end{aligned}$$

These solutions are exact.

Example

Suppose we wish to solve $3x^2 = 5x - 1$.

First we write this in the standard form as $3x^2 - 5x + 1 = 0$ in order to identify the values of a, b and c.

We see that $a = 3$, $b = -5$ and $c = 1$. These values are substituted into the formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 3 \times 1}}{2 \times 3} \\ &= \frac{5 \pm \sqrt{25 - 12}}{6} \\ &= \frac{5 \pm \sqrt{13}}{6} \end{aligned}$$

Again there are two exact solutions. Approximate values could be obtained using a calculator.

Exercise 3

Use the quadratic formula to solve the following quadratic equations.

a) $x^2 - 3x + 2 = 0$ b) $4x^2 - 11x + 6 = 0$ c) $x^2 - 5x - 2 = 0$ d) $3x^2 + 12x + 2 = 0$

e) $2x^2 = 3x + 1$ f) $x^2 + 3 = 2x$ g) $x^2 + 4x = 10$ h) $25x^2 = 40x - 16$

5. Solving quadratic equations by using graphs

In this section we will see how graphs can be used to solve quadratic equations. If the coefficient of x^2 in the quadratic expression $ax^2 + bx + c$ is positive then a graph of $y = ax^2 + bx + c$ will take the form shown in Figure 1(a). If the coefficient of x^2 is negative the graph will take the form shown in Figure 1(b).

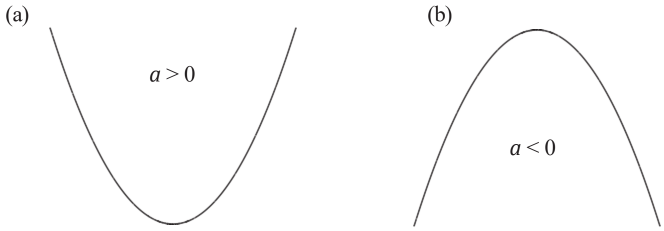


Figure 1. Graphs of $y = ax^2 + bx + c$ have these general shapes

We will now add x and y axes. Figure 2 shows what can happen when we plot a graph of $y = ax^2 + bx + c$ for the case in which a is positive.

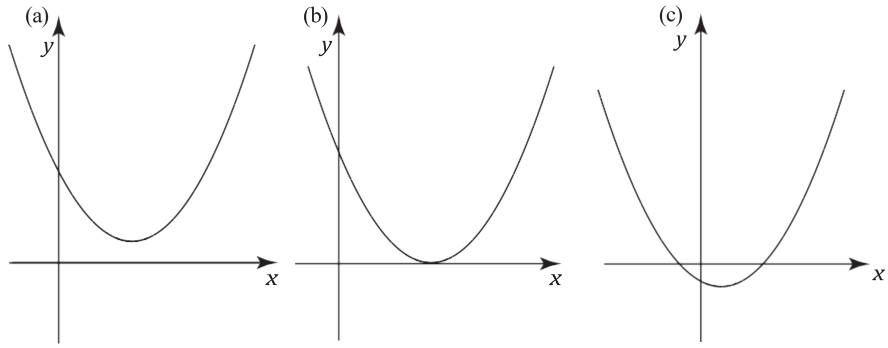


Figure 2. Graphs of $y = ax^2 + bx + c$ when a is positive

The horizontal line, the x axis, corresponds to points on the graph where $y = 0$. So points where the graph touches or crosses this axis correspond to solutions of $ax^2 + bx + c = 0$.

In Figure 2, the graph in (a) never cuts or touches the horizontal axis and so this corresponds to a quadratic equation $ax^2 + bx + c = 0$ having no real roots.

The graph in (b) just touches the horizontal axis corresponding to the case in which the quadratic equation has two equal roots, also called 'repeated roots'.

The graph in (c) cuts the horizontal axis twice, corresponding to the case in which the quadratic equation has two different roots.

What we have done in Figure 2 for the the case in which a is positive we can do for the case in which a is negative. This case is shown in Figure 3.

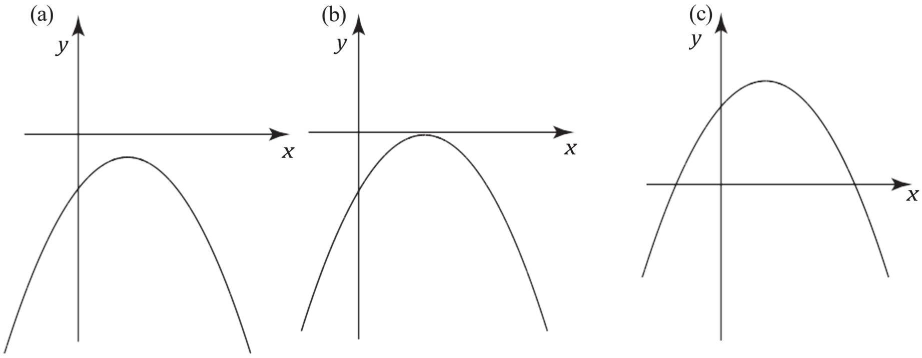


Figure 3. Graphs of $y = ax^2 + bx + c$ when a is negative

Referring to Figure 3: in case (a) there are no real roots. In case (b) there will be repeated roots. Case (c) corresponds to there being two real roots.

Example

Suppose we wish to solve $x^2 - 3x - 2 = 0$.

We consider $y = x^2 - 3x - 2$ and produce a table of values so that we can plot a graph.

x	-2	-1	0	1	2	3	4	5
x^2	4	1	0	1	4	9	16	25
$-3x$	6	3	0	-3	-6	-9	-12	-15
-2	-2	-2	-2	-2	-2	-2	-2	-2
$x^2 - 3x - 2$	8	2	-2	-4	-4	-2	2	8

From this table of values a graph can be plotted, or sketched as shown in Figure 4. From the graph we observe that solutions of the equation $x^2 - 3x - 2 = 0$ lie between -1 and 0, and between 3 and 4.

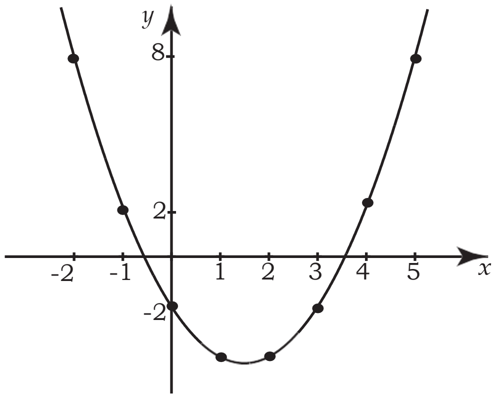


Figure 4. Graph of $y = x^2 - 3x - 2$

Example

We can use the same graph to solve other equations. For example to solve $x^2 - 3x - 2 = 6$ we can simply locate points where the graph crosses the line $y = 6$ as shown in Figure 5.

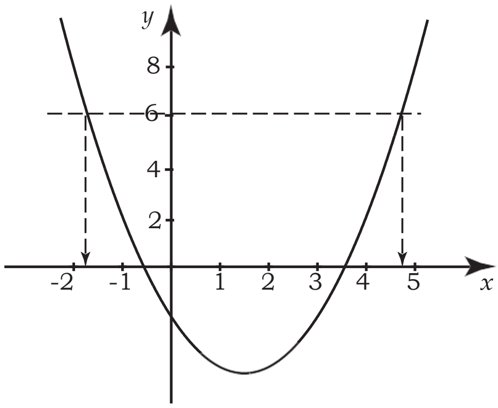


Figure 5. Using the graph of $y = x^2 - 3x - 2$ to solve $x^2 - 3x - 2 = 6$

Example

We can use the same graph to solve $x^2 - 3x - 5 = 0$ by rewriting the equation as $x^2 - 3x - 2 - 3 = 0$ and then as $x^2 - 3x - 2 = 3$. We can then locate points where the graph crosses the line $y = 3$ in order to solve the equation.

Exercise 4

By plotting the graph $y = x^2 - 5x + 2$, solve the equation $x^2 - 5x + 2 = 0$, giving your answers to 1 decimal place.

Use your graph to solve the equations $x^2 - 5x + 2 = 4$, $x^2 - 5x - 1 = 0$, $x^2 - 5x + 2 = 2x$.

Answers

Exercise 1

a) 1, 2 b) $2, -2$ c) 5, -1 d) 0, 5 e) $-4, -15$ f) -2, $\frac{3}{2}$ g) 2, $-\frac{3}{2}$ h) 2, $\frac{3}{4}$

Exercise 2

a) $1, -3$ b) $3 \pm \sqrt{14}$ c) $\frac{5 \pm \sqrt{21}}{2}$ d) $-4 \pm \sqrt{12}$

Exercise 3

a) 1, 2 b) 2, $\frac{3}{4}$ c) $\frac{5 + \sqrt{33}}{2}$ d) $\frac{-12 \pm \sqrt{120}}{6}$ e) $\frac{3 \pm \sqrt{17}}{4}$ f) No real roots g) $-2 \pm \sqrt{14}$ h) $\frac{4}{5}$ repeated.

Exercise 4

a) 4.6, 0.4 b) 5.4, -0.4 c) 5.2, -0.2 d) 6.7, 0.3

Logarithms

mc-TY-logarithms-2009-1

Logarithms appear in all sorts of calculations in engineering and science, business and economics. Before the days of calculators they were used to assist in the process of multiplication by replacing the operation of multiplication by addition. Similarly, they enabled the operation of division to be replaced by subtraction. They remain important in other ways, one of which is that they provide the underlying theory of the logarithm function. This has applications in many fields, for example, the decibel scale in acoustics.

In order to master the techniques explained here it is vital that you do plenty of practice exercises so that they become second nature.

After reading this text and / or viewing the video tutorial on this topic you should be able to:

- explain what is meant by a logarithm
- state and use the laws of logarithms
- solve simple equations requiring the use of logarithms.

Contents

1. Introduction

In this unit we are going to be looking at logarithms. However, before we can deal with logarithms we need to revise indices. This is because logarithms and indices are closely related, and in order to understand logarithms a good knowledge of indices is required.

We know that

$$16 = 2^4$$

Here, the number 4 is the power. Sometimes we call it an exponent. Sometimes we call it an index. In the expression 2^4, the number 2 is called the base.

Example

We know that $64 = 8^2$.

In this example 2 is the power, or exponent, or index. The number 8 is the base.

2. Why do we study logarithms ?

In order to motivate our study of logarithms, consider the following:

we know that $16 = 2^4$. We also know that $8 = 2^3$

Suppose that we wanted to multiply 16 by 8.

One way is to carry out the multiplication directly using long-multiplication and obtain 128. But this could be long and tedious if the numbers were larger than 8 and 16. Can we do this calculation another way using the powers ? Note that

$$16 \times 8 \quad \text{can be written} \quad 2^4 \times 2^3$$

This equals

$$2^7$$

using the rules of indices which tell us to add the powers 4 and 3 to give the new power, 7. What was a multiplication sum has been reduced to an addition sum.

Similarly if we wanted to divide 16 by 8:

$$16 \div 8 \quad \text{can be written} \quad 2^4 \div 2^3$$

This equals

$$2^1 \quad \text{or simply} \quad 2$$

using the rules of indices which tell us to subtract the powers 4 and 3 to give the new power, 1.

If we had a look-up table containing powers of 2, it would be straightforward to look up 2^7 and obtain $2^7 = 128$ as the result of finding 16×8.

Notice that by using the powers, we have changed a multiplication problem into one involving addition (the addition of the powers, 4 and 3). Historically, this observation led John Napier (1550-1617) and Henry Briggs (1561-1630) to develop logarithms as a way of replacing multiplication with addition, and also division with subtraction.

3. What is a logarithm ?

Consider the expression $16 = 2^4$. Remember that 2 is the base, and 4 is the power. An alternative, yet equivalent, way of writing this expression is $\log_2 16 = 4$. This is stated as 'log to base 2 of 16 equals 4'. We see that the logarithm is the same as the power or index in the original expression. It is the base in the original expression which becomes the base of the logarithm.

The two statements

$$16 = 2^4 \qquad \log_2 16 = 4$$

are equivalent statements. If we write either of them, we are automatically implying the other.

Example

If we write down that $64 = 8^2$ then the equivalent statement using logarithms is $\log_8 64 = 2$.

Example

If we write down that $\log_3 27 = 3$ then the equivalent statement using powers is $3^3 = 27$.

So the two sets of statements, one involving powers and one involving logarithms are equivalent. In the general case we have:

Key Point

if $x = a^n$ then equivalently $\log_a x = n$

Let us develop this a little more.

Because $10 = 10^1$ we can write the equivalent logarithmic form $\log_{10} 10 = 1$.
Similarly, the logarithmic form of the statement $2^1 = 2$ is $\log_2 2 = 1$.

In general, for any base a, $a = a^1$ and so $\log_a a = 1$.

Key Point

$$\log_a a = 1$$

We can see from the Examples above that indices and logarithms are very closely related. In the same way that we have rules or laws of indices, we have laws of logarithms. These are developed in the following sections.

4. Exercises

1. Write the following using logarithms instead of powers

a)	$8^2 = 64$	b)	$3^5 = 243$	c)	$2^{10} = 1024$	d)	$5^3 = 125$
e)	$10^6 = 1000000$	f)	$10^{-3} = 0.001$	g)	$3^{-2} = \frac{1}{9}$	h)	$6^0 = 1$
i)	$5^{-1} = \frac{1}{5}$	j)	$\sqrt{49} = 7$	k)	$27^{2/3} = 9$	l)	$32^{-2/5} = \frac{1}{4}$

2. Determine the value of the following logarithms

a)	$\log_3 9$	b)	$\log_2 32$	c)	$\log_5 125$	d)	$\log_{10} 10000$
e)	$\log_4 64$	f)	$\log_{25} 5$	g)	$\log_8 2$	h)	$\log_{81} 3$
i)	$\log_3\left(\frac{1}{27}\right)$	j)	$\log_7 1$	k)	$\log_8\left(\frac{1}{8}\right)$	l)	$\log_4 8$
m)	$\log_a a^5$	n)	$\log_c \sqrt{c}$	o)	$\log_s s$	p)	$\log_e\left(\frac{1}{e}\right)$

5. The first law of logarithms

Suppose

$$x = a^n \quad \text{and} \quad y = a^m$$

then the equivalent logarithmic forms are

$$\log_a x = n \quad \text{and} \quad \log_a y = m \qquad (1)$$

Using the first rule of indices

$$xy = a^n \times a^m = a^{n+m}$$

Now the logarithmic form of the statement $xy = a^{n+m}$ is $\log_a xy = n + m$. But $n = \log_a x$ and $m = \log_a y$ from (1) and so putting these results together we have

$$\log_a xy = \log_a x + \log_a y$$

So, if we want to multiply two numbers together and find the logarithm of the result, we can do this by adding together the logarithms of the two numbers. This is the first law.

Key Point

$$\log_a xy = \log_a x + \log_a y$$

6. The second law of logarithms

Suppose $x = a^n$, or equivalently $\log_a x = n$. Suppose we raise both sides of $x = a^n$ to the power m:

$$x^m = (a^n)^m$$

Using the rules of indices we can write this as

$$x^m = a^{nm}$$

Thinking of the quantity x^m as a single term, the logarithmic form is

$$\log_a x^m = nm = m \log_a x$$

This is the second law. It states that when finding the logarithm of a power of a number, this can be evaluated by multiplying the logarithm of the number by that power.

Key Point

$$\log_a x^m = m \log_a x$$

7. The third law of logarithms

As before, suppose

$$x = a^n \qquad \text{and} \qquad y = a^m$$

with equivalent logarithmic forms

$$\log_a x = n \qquad \text{and} \qquad \log_a y = m \tag{2}$$

Consider $x \div y$.

$$\begin{aligned} \frac{x}{y} &= a^n \div a^m \\ &= a^{n-m} \end{aligned}$$

using the rules of indices.

In logarithmic form

$$\log_a \frac{x}{y} = n - m$$

which from (2) can be written

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

This is the third law.

Key Point

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

8. The logarithm of 1

Recall that any number raised to the power zero is 1: $a^0 = 1$. The logarithmic form of this is

$$\log_a 1 = 0$$

Key Point

$$\log_a 1 = 0$$

The logarithm of 1 in any base is 0.

9. Examples

Example

Suppose we wish to find $\log_2 512$.

This is the same as being asked 'what is 512 expressed as a power of 2 ?'
Now 512 is in fact 2^9 and so $\log_2 512 = 9$.

Example

Suppose we wish to find $\log_8 \frac{1}{64}$.

This is the same as being asked 'what is $\frac{1}{64}$ expressed as a power of 8 ?'

Now $\frac{1}{64}$ can be written 64^{-1}. Noting also that $8^2 = 64$ it follows that

$$\frac{1}{64} = 64^{-1} = (8^2)^{-1} = 8^{-2}$$

using the rules of indices. So $\log_8 \frac{1}{64} = -2$.

Example

Suppose we wish to find $\log_5 25$.

This is the same as being asked 'what is 25 expressed as a power of 5 ?'

Now $5^2 = 25$ and so $\log_5 25 = 2$.

Example

Suppose we wish to find $\log_{25} 5$.

This is the same as being asked 'what is 5 expressed as a power of 25 ?'

We know that 5 is a square root of 25, that is $5 = \sqrt{25}$. So $25^{\frac{1}{2}} = 5$ and so $\log_{25} 5 = \frac{1}{2}$.

Notice from the last two examples that by interchanging the base and the number

$$\log_{25} 5 = \frac{1}{\log_5 25}$$

This is true more generally:

Key Point

$$\log_b a = \frac{1}{\log_a b}$$

To illustrate this again, consider the following example.

Example

Consider $\log_2 8$. We are asking 'what is 8 expressed as a power of 2 ?' We know that $8 = 2^3$ and so $\log_2 8 = 3$.

What about $\log_8 2$? Now we are asking 'what is 2 expressed as a power of 8 ?' Now $2^3 = 8$ and so $2 = \sqrt[3]{8}$ or $8^{1/3}$. So $\log_8 2 = \frac{1}{3}$.

We see again

$$\log_8 2 = \frac{1}{\log_2 8}$$

10. Exercises

3 Each of the following expressions can be simplified to $\log N$.
Determine the value of N in each case. We have not explicitly written down the base. You can assume the base is 10, but the results are identical whichever base is used.

a) $\log 3 + \log 5$
b) $\log 16 - \log 2$
c) $3 \log 4$
d) $2 \log 3 - 3 \log 2$
e) $\log 236 + \log 1$
f) $\log 236 - \log 1$
g) $5 \log 2 + 2 \log 5$
h) $\log 128 - 7 \log 2$
i) $\log 2 + \log 3 + \log 4$
j) $\log 12 - 2 \log 2 + \log 3$
k) $5 \log 2 + 4 \log 3 - 3 \log 4$
l) $\log 10 + 2 \log 3 - \log 2$

11. Standard bases

There are two bases which are used much more commonly than any others and deserve special mention. These are

base 10 and base e

Logarithms to base 10, $\log_{10}$, are often written simply as $\log$ without explicitly writing a base down. So if you see an expression like $\log x$ you can assume the base is 10. Your calculator will be pre-programmed to evaluate logarithms to base 10. Look for the button marked log.

The second common base is e. The symbol e is called the exponential constant and has a value approximately equal to 2.718. This is a number like π in the sense that it has an infinite decimal expansion. Base e is used because this constant occurs frequently in the mathematical modelling of many physical, biological and economic applications. Logarithms to base e, $\log_e$, are often written simply as $\ln$. If you see an expression like $\ln x$ you can assume the base is e. Such logarithms are also called Naperian or natural logarithms. Your calculator will be pre-programmed to evaluate logarithms to base e. Look for the button marked ln.

Key Point

Common bases:

$\log$ means $\log_{10}$ $\qquad$ $\ln$ means $\log_e$

where e is the exponential constant.
Useful results:

$$\log 10 = 1, \qquad \ln e = 1$$

12. Using logarithms to solve equations

We can use logarithms to solve equations where the unknown is in the power.

Suppose we wish to solve the equation $3^x = 5$. We can solve this by taking logarithms of both sides. Whilst logarithms to any base can be used, it is common practice to use base 10, as these are readily available on your calculator. So,

$$\log 3^x = \log 5$$

Now using the laws of logarithms, the left hand side can be re-written to give

$$x \log 3 = \log 5$$

This is more straightforward. The unknown is no longer in the power. Straightaway

$$x = \frac{\log 5}{\log 3}$$

If we wanted, this value can be found from a calculator.

Example

Solve $3^x = 5^{x-2}$. Again, notice that the unknown appears in the power. Take logs of both sides.

$$\log 3^x = \log 5^{x-2}$$

Now use the laws of logarithms.

$$x \log 3 = (x-2)\log 5$$

Notice now that the x we are trying to find is no longer in a power. Multiplying out the brackets

$$x \log 3 = x \log 5 - 2\log 5$$

Rearrange this equation to get the two terms involving x on one side and the remaining term on the other side.

$$2\log 5 = x\log 5 - x\log 3$$

Factorise the right-hand side by extracting the common factor of x.

$$\begin{aligned} 2\log 5 &= x(\log 5 - \log 3) \\ &= x \log \frac{5}{3} \end{aligned}$$

using the laws of logarithms.

And finally

$$x = \frac{2\log 5}{\log\left(\frac{5}{3}\right)}$$

If we wanted, this value can be found from a calculator.

13. Inverse operations

Suppose we pick a base, 2 say.

Suppose we pick a power, 8 say.

We will now raise the base 2 to the power 8, to give 2^8.

Suppose we now take logarithms to base 2 of 2^8.

We then have

$$\log_2 2^8$$

Using the laws of logarithms we can write this as

$$8 \log_2 2$$

Recall that $\log_a a = 1$, so $\log_2 2 = 1$, and so we have simply 8 again, the number we started with.

So, raising the base 2 to a power, and then finding the logarithm to base 2 of the result are inverse operations.

Let's look at this another way.

Suppose we pick a number, 8 say.

Suppose we find its logarithm to base 2, to evaluate $\log_2 8$.

Suppose we now raise the base 2 to this power: $2^{\log_2 8}$.

Because $8 = 2^3$ we can write this as $2^{\log_2 2^3}$. Using the laws of logarithms this equals $2^{3 \log_2 2}$ which equals 2^3 or 8, since $\log_2 2 = 1$. We see that raising the base 2 to the logarithm of a number to base 2 results in the original number.

So raising a base to a power, and finding the logarithm to that base are inverse operations. Doing one operation, and then following it by the other, we end up where we started.

Example

Suppose we are working in base e. We can pick a number x and evaluate e^x. If we follow this by taking logarithms to base e we obtain

$$\ln e^x$$

Using the laws of logarithms this equals

$$x \ln e$$

but $\ln e = 1$ and so we are left with simply x again. So, raising the base e to a power, and then finding logarithms to base e are inverse operations.

Example

Suppose we are working in base 10. We can pick a number x and evaluate 10^x. If we follow this by taking logarithms to base 10 we obtain

$$\log 10^x$$

Using the laws of logarithms this equals

$$x \log 10$$

but $\log 10 = 1$ and so we are left with simply x again. So, raising the base 10 to a power, and then finding logarithms to base 10 are inverse operations.

Key Point

$$\ln e^x = x, \qquad e^{\ln x} = x$$

Similarly,

$$\log 10^x = x, \qquad 10^{\log x} = x$$

These results will be useful in doing calculus, especially in solving differential equations.

14. Exercises

4 Use logarithms to solve the following equations

a) $10^x = 5$ b) $e^x = 8$ c) $10^x = \frac{1}{2}$ d) $e^x = 0.1$

e) $4^x = 12$ f) $3^x = 2$ g) $7^x = 1$ h) $\left(\frac{1}{2}\right)^x = \frac{1}{100}$

i) $\pi^x = 10$ j) $e^x = \pi$ k) $\left(\frac{1}{3}\right)^x = 2$ l) $10^x = e^{2x-1}$

Answers to Exercises on Logarithms

1. a) $\log_8 64 = 2$ b) $\log_3 243 = 5$ c) $\log_2 1024 = 10$

   d) $\log_5 125 = 3$ e) $\log_{10} 1000000 = 6$ f) $\log_{10} 0.001 = -3$

   g) $\log_3 \left(\frac{1}{9}\right) = -2$ h) $\log_6 1 = 0$ i) $\log_5 \left(\frac{1}{5}\right) = -1$

   j) $\log_{49} 7 = \frac{1}{2}$ k) $\log_{27} 9 = \frac{2}{3}$ l) $\log_{32} \frac{1}{4} = -\frac{2}{5}$

2. a) 2 b) 5 c) 3 d) 4

   e) 3 f) $\frac{1}{2}$ g) $\frac{1}{3}$ h) $\frac{1}{4}$

   i) -3 j) 0 k) -1 l) $\frac{3}{2}$

   m) 5 n) $\frac{1}{2}$ o) 1 p) -3

3. a) 15 b) 8 c) 64 d) $\frac{9}{8}$

   e) 236 f) 236 g) 800 h) 1

   i) 24 j) 9 k) $\frac{2592}{64} = \frac{81}{2}$ l) 45

4. All answers are correct to 3 decimal places

   a) 0.699 b) 2.079 c) -0.301 d) -2.303

   e) 1.792 f) 0.631 g) 0 h) 6.644

   i) 2.011 j) 1.145 k) -0.631 l) -3.305

Pythagoras' theorem

mc-TY-pythagoras-2009-1

Pythagoras' theorem is well-known from schooldays. In this unit we revise the theorem and use it to solve problems involving right-angled triangles. We will also meet a less-familiar form of the theorem.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- state Pythagoras' theorem
- use Pythagoras' theorem to solve problems involving right-angled triangles.

Contents

1. Introduction

The Theorem of Pythagoras is a well-known theorem. It is also a very old one, not only does it bear the name of Pythagoras, an ancient Greek, but it was also known to the ancient Babylonians and to the ancient Egyptians. Most school students learn of it as $a^2 + b^2 = c^2$. The actual statement of the theorem is more to do with areas. So, let's have a look at the statement of the theorem.

2. The Theorem of Pythagoras

The theorem makes reference to a right-angled triangle such as that shown in Figure 1. The side opposite the right-angle is the longest side and is called the hypotenuse.

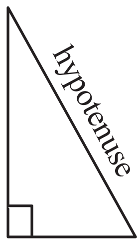


Figure 1. A right-angled triangle with hypotenuse shown.

What the theorem says is that the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the two shorter sides. Figure 2 shows squares drawn on the hypotenuse and on the two shorter sides. The theorem tells us that area A + area B = area C.

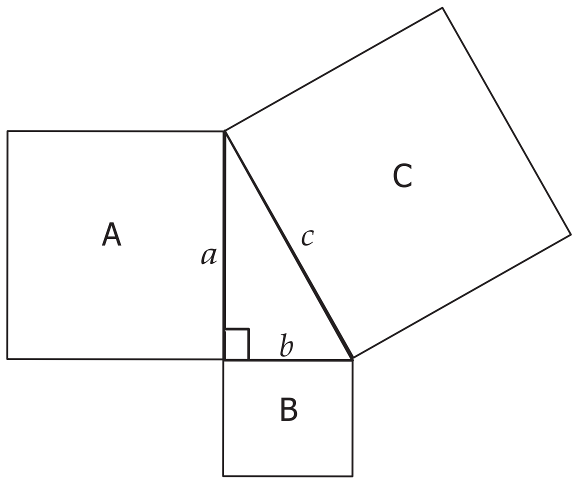


Figure 2. A right-angled triangle with squares drawn on each side.

An excellent demonstration of this is available on the accompanying video. If we denote the lengths of the sides of the triangle as a, b and c, as shown, then area A = a^2, area B = b^2 and area C = c^2. So, using Pythagoras' theorem

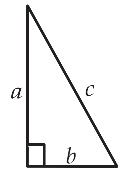
$$\begin{aligned} \text{area A} + \text{area B} &= \text{area C} \\ a^2 + b^2 &= c^2 \end{aligned}$$

This is the traditional result.

Key Point

Pythagoras' theorem:

$$a^2 + b^2 = c^2$$



Example

Suppose we wish to find the length of the hypotenuse of the right-angled triangle shown in Figure 4. We have labelled the hypotenuse c.

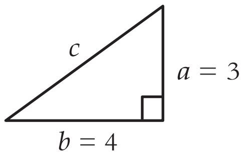


Figure 4.

Using the theorem:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= c^2 \\ 9 + 16 &= c^2 \\ 25 &= c^2 \\ 5 &= c \end{aligned}$$

So 5 is the length of the hypotenuse, the longest side of the triangle.

Example

In this Example we will assume we know the length of the hypotenuse.

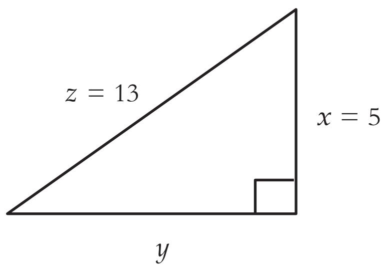


Figure 5.

The corresponding statement of Pythagoras' theorem is $x^2 + y^2 = z^2$. So,

$$\begin{aligned} x^2 + y^2 &= z^2 \\ 5^2 + y^2 &= 13^2 \\ 25 + y^2 &= 169 \\ y^2 &= 144 \\ y &= \sqrt{144} = 12 \end{aligned}$$

So, 12 is the length of the unknown side.

Example

Suppose we wish to find the length q in Figure 6. The statement of the theorem is now

$$p^2 + q^2 = r^2$$

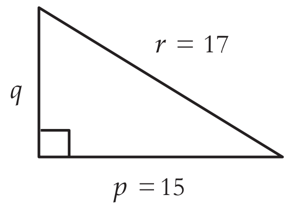


Figure 6.

$$\begin{aligned} p^2 + q^2 &= r^2 \\ 15^2 + q^2 &= 17^2 \\ 225 + q^2 &= 289 \\ q^2 &= 64 \\ q &= 8 \end{aligned}$$

So, 8 is the length of the unknown side.

In each of these three Examples, the answers have been exact and they have been whole number values. These whole number triples, ((3,4,5), (5,12,13), (8,15,17)), or Pythagorean triples, as we call them, occur quite frequently. When you do questions like these, you probably won't be so lucky to get exact answers as we have done here; you almost certainly will have to use a calculator for many of them and you have to decide to approximate the answer to a given number of decimal places or a given number of significant figures.

Exercise 1

1. Determine, to 2 decimal places, the length of the hypotenuses of the right-angled triangles whose two shorter sides have the lengths given below.

   a) 5 cm, 12 cm b) 1 cm, 2 cm c) 3 cm, 4 cm

   d) 1 cm, 1 cm e) 1.73 cm, 1 cm f) 2 cm, 5 cm

2. Determine, to 2 decimal places, the length of the third sides of the right-angled triangles where the hypotenuse and other side have the lengths given below.

   a) 8 cm, 2 cm b) 5 cm, 4 cm c) 2 cm, 1 cm

   d) 6 cm, 5 cm e) 10 cm, 7 cm f) 1 cm, 0.5 cm

3. A further application of the theorem

Let's have a look at another application of Pythagoras' theorem.
Look at the cuboid shown in Figure 7. Suppose we wish to find the length, y, of the diagonal of this cuboid. This is the bold line in Figure 7. Note that ABC is a right-angled triangle with the right-angle at C. Note also, that ACD is a right-angled triangle with hypotenuse AC. Let AC have length x, as shown.

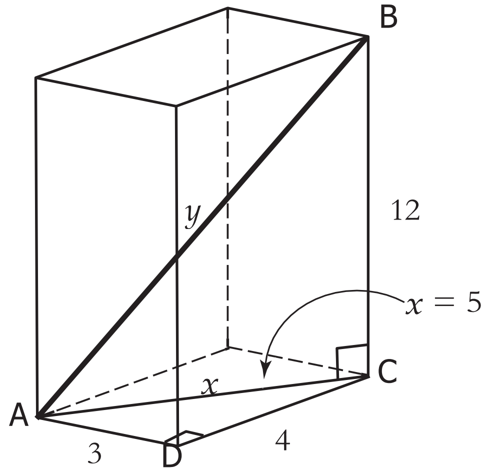


Figure 7.

Referrring to triangle ACD and using Pythagoras' theorem:

$$\begin{aligned} 3^2 + 4^2 &= x^2 \\ 9 + 16 &= x^2 \\ x^2 &= 25 \\ x &= 5 \end{aligned}$$

Now let the length of the diagonal AB be y. Using Pythagoras' theorem in triangle ABC:

$$\begin{aligned} 5^2 + 12^2 &= y^2 \\ 25 + 144 &= y^2 \\ 169 &= y^2 \\ y &= \sqrt{169} = 13 \end{aligned}$$

So we can use the theorem of Pythagoras in 3-dimensions; we can use it to solve problems that are set up in 3-dimensional objects.

Exercise 2

1. Calculate the length of the diagonals of the following cuboids
   a) $2 \times 3 \times 4$ b) $1 \times 1 \times 3$ c) 5×5 5
   Give your×answers to 2 decimal places.

4. Applications in cartesian geometry

Cartesian geometry is geometry that is set out on a plane that uses cartesian co-ordinates. These are the familiar (x, y) coordinates you will have seen before. We have already seen that Pythagoras' theorem gives us a relationship which is satisfied between the lengths of the sides of a right-angled triangle. However, we can turn this theorem around. If we study any triangle, and find that the area of the square on the longest side is equal to the sum of the areas of the squares on the two shorter sides of the triangle, then the triangle must be right-angled. So we can use Pythagoras' theorem to tell whether a triangle is right-angled or not.

Example

Suppose we have the co-ordinates of three points:

$$(3, 4) \quad (2, 6) \quad (1, 0)$$

Can we use Pythagoras' theorem to find out whether they form the corners of a right-angled triangle ? The points are plotted in Figure 8. Looking at Figure 8 we might guess that the triangle does contain a right-angle, but we can't be sure. The scales on the two axes are not quite the same and so appearances can be deceptive.

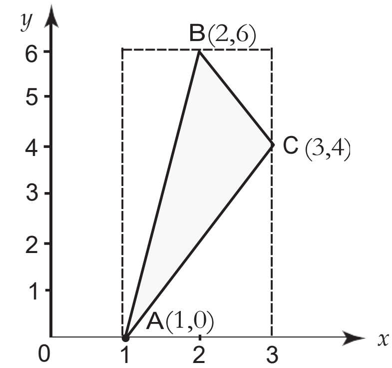


Figure 8.

From Figure 8 note that

$$\begin{aligned}(AB)^2 &= 1^2 + 6^2 = 1 + 36\\ &= 37\\ (BC)^2 &= 2^2 + 1^2 = 4 + 1\\ &= 5\\ \\ (AC)^2 &= 2^2 + 4^2 = 4 + 16\\ &= 20\end{aligned}$$

Now ask the question. Do the squares of the shorter sides add together to give the square of the longer side ?

$$(AC)^2 + (BC)^2 = 5 + 20 = 25 \neq 37$$

So we do not have a right-angled triangle. In fact because the longest side, AB, is greater than that which would be required to form a right-angle (i.e. $\sqrt{25} = 5$) we can deduce that the angle at C is in fact greater than 90°. Thus C is an obtuse angle. This is not immediately obvious from a sketch.

Exercise 3

1. The lengths of the sides of a number of different triangles are given below. In each case, determine whether the largest angle is obtuse, right angle or acute.

   a) 1, 1, 1 b) 1, 1, 2 c) 4, 3, 2 d) 5, 7, 9

   e) 5, 12, 13 f) 2, 6, 4 g) 8, 10, 6 h) 1, 2, 2

2. The co-ordinates of the vertices of a number of triangles are given below. In each case, determine if the triangle is right angled.

   a) (1,2), (2,3), (5,0) b) (2,1), (4,2), (2,6) c) (3,2), (5,1), (2,5)

5. A final result

Finally, the fact that Pythagoras' Theorem is about squares is fairly well-known. What is not so well-known, although it is fairly obvious once you have seen it, is that if you take any regular figure or similar figures and place them on the sides of a right-angled triangle, then the area of the figure on the hypotenuse is equal to the sum of the areas of the figures on the two shorter sides. Consider, for example, Figure 9 where we have drawn semi-circles on each side of the right-angled triangle.

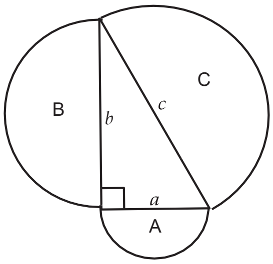


Figure 9. Semi-circles drawn on the sides of the right-angled triangle.

From the formula for the area of a circle:

$$\text{Area of } A = \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 = \frac{\pi}{8}a^2$$

$$\text{Area of } B = \frac{1}{2}\pi\left(\frac{b}{2}\right)^2 = \frac{\pi}{8}b^2$$

$$\text{Area of } C = \frac{1}{2}\pi\left(\frac{c}{2}\right)^2 = \frac{\pi}{8}c^2$$

and so

$$\begin{aligned}\text{Area of } A + \text{Area of } B &= \frac{\pi}{8}a^2 + \frac{\pi}{8}b^2 \\ &= \frac{\pi}{8}(a^2 + b^2)\end{aligned}$$

But from Pythagoras' theorem $a^2 + b^2 = c^2$ and so

$$\begin{aligned}\text{Area of } A + \text{Area of } B &= \frac{\pi}{8}c^2 \\ &= \text{Area of } C\end{aligned}$$

and so it is true that $\text{Area of } A + \text{Area of } B = \text{Area of } C$.

Answers

Exercise 1

1. a) 13 cm b) 2.24 cm c) 5 cm d) 1.41 cm e) 2.00 cm f) 5.39 cm
2. a) 7.75 cm b) 3 cm c) 1.73 cm d) 3.32 cm e) 7.14 cm f) 0.87 cm

Exercise 2

1. a) 5.39 b) 3.32 c) 8.66

Exercise 3

1. a) acute b) obtuse c) obtuse d) obtuse e) right angle f) obtuse g) right angle h) acute
2. a) Yes b) Yes c) No

Trigonometrical ratios in a right-angled triangle

mc-TY-trigratios-2009-1

Knowledge of the trigonometrical ratios sine, cosine and tangent, is vital in very many fields of engineering, mathematics and physics. This unit introduces them and provides examples of how they can be used in the solution of problems.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- define the ratios sine, cosine and tangent with reference to a right-angled triangle.
- use the trig ratios to solve problems involving triangles.
- quote trig ratios for commonly occuring angles.

Contents

1. Introduction

In this unit we are going to be having a look at trigonometrical ratios in a right-angled triangle. We will usually refer to these as 'trig ratios' for short.

2. Introducing the tangent ratio

Study the diagram in Figure 1. On the horizontal line we have marked the points O, A_1, A_2 and A_3, and each of these points is 10 cm apart. We have drawn vertical lines from the points A_1, A_2 and A_3 to form right-angled triangles.

There is an angle marked at O, and this angle remains the same even though the separation of the lines OA_3 and OB_3 increases as we move away from O.

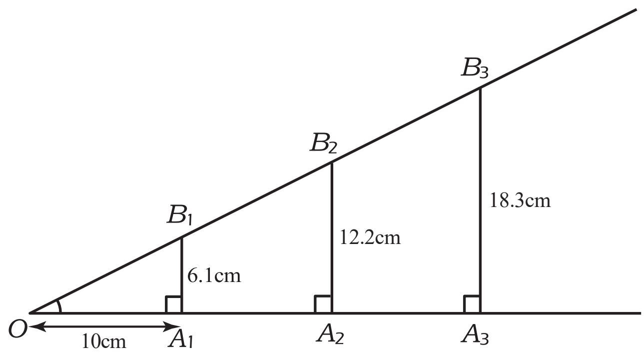


Figure 1. The angle marked at O remains the same as we move further from O.

We want to study the ratio of A_1B_1 to OA_1, of A_2B_2 to OA_2 and of A_3B_3 to OA_3.
Using a ruler we have measured A_1B_1 and found it to be 6.1 cm.
So

$$\frac{A_1B_1}{OA_1} = \frac{6.1}{10} = 0.61$$

Similarly, we have measured A_2B_2 and found it to be 12.2 cm. So

$$\frac{A_2B_2}{OA_2} = \frac{12.2}{20} = 0.61$$

Finally, we have measured A_3B_3 and found it to be 18.3 cm. So

$$\frac{A_3B_3}{OA_3} = \frac{18.3}{30} = 0.61$$

We see that all of the three ratios are the same. So if we take the length of the side in a right-angled triangle which is *opposite* to an angle and we divide it by the length of the side which is *adjacent* to it, then the ratio

$$\frac{\text{OPPOSITE}}{\text{ADJACENT}} = \text{constant for the given angle}$$

We have a name for this ratio. We call it the tangent of the angle.

3. Labelling the sides of a right-angled triangle

Consider Figure 2.

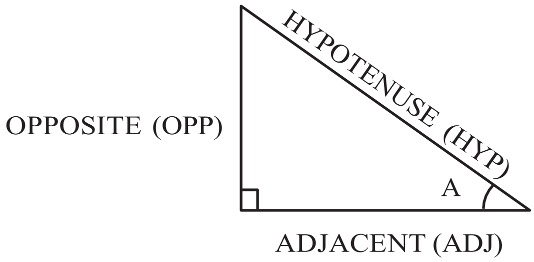


Figure 2. A right-angled triangle with angle A marked.

The longest side of a right-angled triangle is always called the hypotenuse, usually shortened to HYP. This side is always opposite the right-angle. The side opposite angle A has been labelled OPP, and the remaining side, which is adjacent to A has been labelled ADJ.

Notice that if we look at a different angle, some of these quantities change. Consider Figure 3.

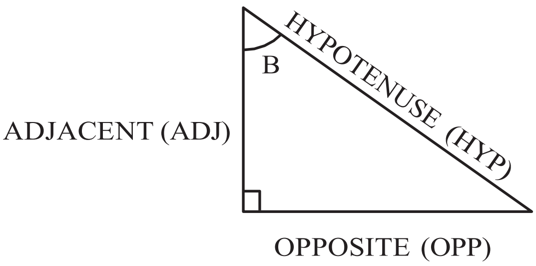


Figure 2. A right-angled triangle with angle B marked.

The hypotenuse is as it was in Figure 2, but the other two labels have changed.

4. The sine, cosine and tangent ratios

Referring to Figure 4, recall that we have already named the ratio $\frac{\text{OPPOSITE}}{\text{ADJACEN}}$ as the tangent of the given angle. We usually shorten this to simply tan. So

$$\text{tangent } A = \tan A = \frac{\text{OPP}}{\text{ADJ}}$$

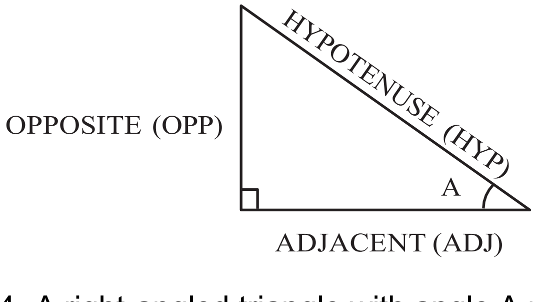


Figure 4. A right-angled triangle with angle A marked.

We can also calculate some other ratios from this triangle.

The ratio $\frac{\text{OPP}}{\text{HYP}}$ is called the sine of A which we abbreviate to $\sin A$.

The ratio $\frac{\text{ADJ}}{\text{HYP}}$ is called the cosine of A which we abbreviate to $\cos A$.

All of these ratios have already been worked out and are available in published tables. Before the days of calculators mathematicians used to work these out quite regularly and publish books of tables of the sines, cosines and tangents of all the angles through from 0 to 90°. Nowadays a calculator is invaluable and you really do need one for this sort of work.

5. Remembering the definitions

It will help if you have a way of recalling these definitions. One of these ways is by remembering a nonsense word:

SOH TOA CAH

sine is opposite over hypotenuse,
tangent is opposite over adjacent
and cosine is adjacent over hypotenuse.

Some people remember it as

SOH CAH TOA

simply changing the syllables around.

Others remember it by a little verse:

Tom's Old Aunt (TOA)
Sat On Him (SOH)
Cursing At Him. (CAH)

Whichever you learn, it will be helpful in order to remember these ratios.

Key Point

$$\sin A = \frac{\text{OPPOSITE}}{\text{HYPOTENUSE}}, \quad \cos A = \frac{\text{ADJACENT}}{\text{HYPOTENUSE}}, \quad \tan A = \frac{\text{OPPOSITE}}{\text{ADJACENT}}$$

6. Examples

Example

Many of the examples that we want to look at actually come from the very practical area of surveying - the problem of finding out the size of something, or the length of something, when you cannot actually measure it, perhaps the height of a tower. So suppose we want to know the height of the tower in Figure 5.

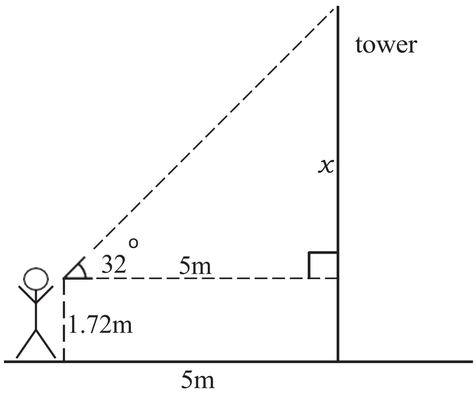


Figure 5.

It is possible to measure the angle between the horizontal and a line from a surveying instrument to the top of the tower. Suppose this angle has been found to be 32° as shown. It is also straightforward to measure how far away we are standing from the base of the tower. Suppose this is 5m. Suppose the height of the person doing the surveying is 1.72m. So how high is the tower ?

Observe the right-angled triangle in Figure 6. The side we wish to find is opposite the angle of 32° as shown in Figure 6. We know the adjacent side is 5m. So we ask 'what trig ratio links opposite and adjacent ?' The answer is the tangent ratio.

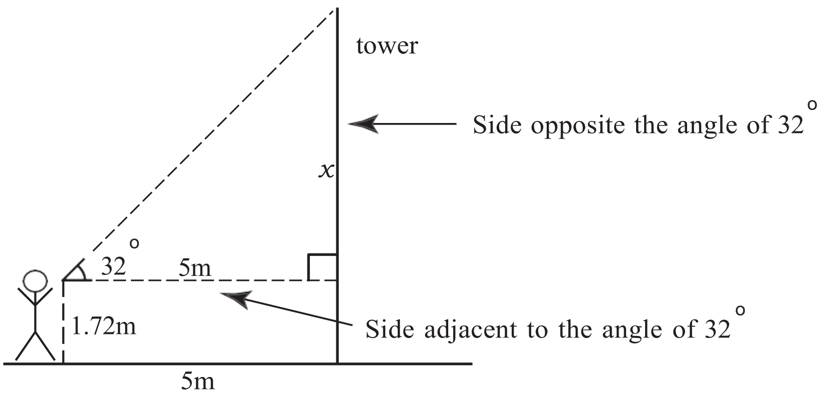


Figure 6.

Let the length of the opposite side be x. Then

$$
\begin{aligned}
\frac{\text{OPP}}{\text{ADJ}} &= \tan 32^\circ \\
\frac{x}{5} &= \tan 32^\circ \\
x &= 5 \times \tan 32^\circ
\end{aligned}
$$

We now need a calculator to evaluate $\tan 32^\circ$. You should check for yourself that $5 \times \tan 32^\circ =$ 3.124. One word of warning: most calculators operate with angles measured in one of two ways, degrees or radians. You need to make sure the calculator is operating with the correct measurement of the angle, in this case, degrees. So, continuing,

$$\begin{aligned} x &= 5 \times \tan 32^\circ \\ &= 3.124 \end{aligned}$$

Finally the height of the tower can be found by adding on the height of the surveyor:

$$\begin{aligned} \text{height} &= 3.124 + 1.72 \\ &= 4.844 \text{ m} \\ &= 4.8 \text{ m} \quad (2 \text{ s.f.}) \end{aligned}$$

So, the height of the tower is 4.8m.

Exercise 1

1. The angle of elevation of the top of a tree from a point on the ground 10 m from the base of the tree is 28°. What is the height of the tree (to 1 decimal place)?

2. Using a surveying instrument 1.6 m high, the angle of elevation of the top of a tower is measured to be 55° from a point 6 m from the base of the tower. How high is the tower (to 1 decimal place)?

3. The angle of elevation of the top of a 20 m high mast from a point at ground level is 34°. How far is the point from the foot of the mast (to 1 decimal place)?

Example

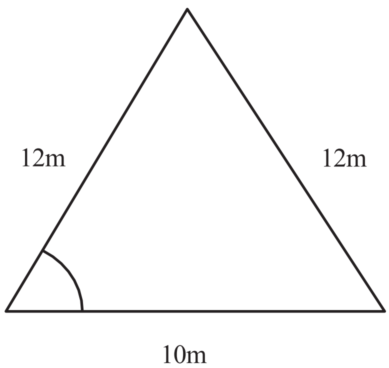


Figure 7.

In this Example consider the isosceles triangle shown in Figure 7. Suppose we wish to find the marked angle. In this Example there does not appear to be a right-angle. In questions like this you must look to introduce a right-angle for yourself. If we divide the isosceles triangle in half we can form a right-angle as shown in Figure 8. We have labelled the required angle A. From studying Figure 8 you will see that we know the hypotenuse and the adjacent side of the right-angled triangle and so we use the cosine ratio.

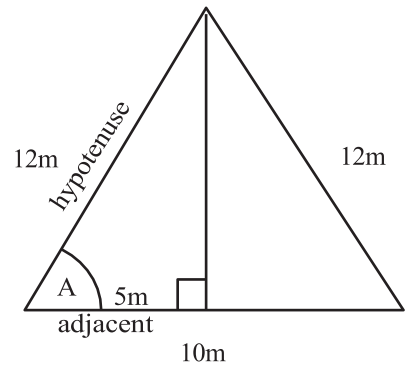


Figure 8.

$$\cos A = \frac{\text{ADJ}}{\text{HYP}} = \frac{5}{12}$$

So the angle we want, A, is an angle which has a cosine equal to $\frac{5}{1}$. We have a way of writing this:

$$A = \cos^{-1}\frac{5}{12}$$

This is read as 'A is the angle whose cosine is $\frac{5}{1}$'. Note that the -1 does not denote a power. Your calculator will have a button marked $\cos^{-1}$ in order to do this calculation. Check that you can use it correctly to find

$$\begin{aligned} A &= \cos^{-1}\frac{5}{12} \\ &= 65.3757^\circ \\ &= 65.4^\circ \text{ (1 d.p.)} \end{aligned}$$

So, the required angle has been shown to be 65.4°.

Example

Suppose we wish to find the height of the isosceles triangle in Figure 9. In this example we want to know the length, h say, of the side opposite the angle of 72°, and we know the length of the hypotenuse. The ratio which links the opposite and the hypotenuse is the sine.

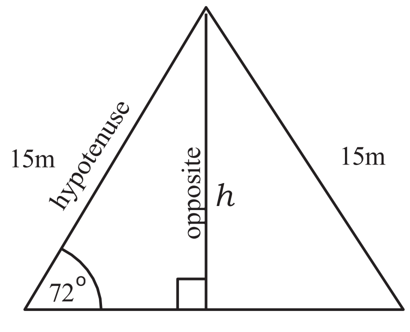


Figure 9.

$$
\begin{aligned}
\frac{\text{OPP}}{\text{HYP}} &= \sin 72^\circ \\
\frac{h}{15} &= \sin 72^\circ \\
h &= 15 \times \sin 72^\circ \\
&= 14.3 \text{ m} \quad (3 \text{ s.f.})
\end{aligned}
$$

So we have shown that the height of the triangle is 14.3m.

Exercise 2

1. An isosceles triangle has base 8 cm and sloping sides both with length 10 cm. What is the base angle of this triangle (to the nearest degree)?
2. A supporting cable of length 30 m is fastened to the top of a 20 m high mast. What angle does the cable make with the ground? How far away from the foot of the mast is it anchored to the ground (to 1 decimal place)?
3. A right-angled triangle has sides 5, 12, 13. What is the size of the smallest angle in this triangle (to the nearest degree)?

Example
Consider the right-angled triangle shown in Figure 10. Suppose we wish to find the length of the hypotenuse.

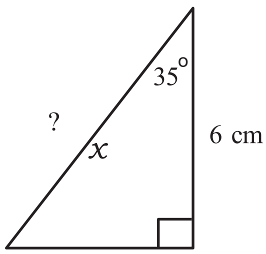


Figure 10.

We know the adjacent side is 6 cm. We want to know the hypotenuse, x, say. The ratio which links these is the cosine.

$$
\begin{aligned}
\frac{\text{ADJ}}{\text{HYP}} &= \cos 35^\circ \\
\frac{6}{x} &= \cos 35^\circ \\
6 &= x \cos 35^\circ \\
\frac{6}{\cos 35^\circ} &= x \\
x &= 7.3246 \\
&= 7.32 \text{ m} \quad (3 \text{ s.f.})
\end{aligned}
$$

Exercise 3

1. What is the height (to 1 decimal place) of an isosceles triangle with base angle 65^o and sloping sides with length 10 cm? What is the length of the base of this triangle (to 1 decimal place)?

2. One angle in a right angled triangle is 50^o and the side opposite this angle has length 5 cm. What is the length of the hypotenuse (to 1 decimal place)?

3. In a right angled triangle, the hypotenuse has length 8 m and one angle is 55^o. What is the length of the shortest side (to 1 decimal place)?

7. Some common angles and their trig ratios

One of the things that we need to have a look at are some quite specific angles and their sines, cosines and tangents. It is important that these are learnt - this is because they are exact and because mathematicians, scientists and engineers use these a great deal in their work.

The angles that we are talking about are 0°, 30°, 45°, 60° and 90°.

Consider Figure 11 which shows a right-angled isosceles triangle with angles of 45°, 45°, and 90°.

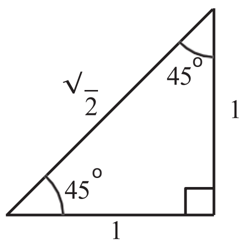


Figure 11.

From this triangle we can deduce

$$\sin 45^\circ = \frac{1}{\sqrt{2}}, \quad \cos 45^\circ = \frac{1}{\sqrt{2}}, \quad \tan 45^\circ = \frac{1}{1} = 1$$

Now consider the equilateral triangle shown in Figure 12. All sides have length 2. All angles will be 60°.

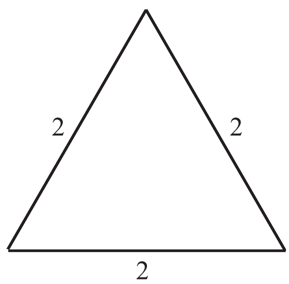


Figure 12.

If we split the triangle in half we can introduce angles of 30° as shown in Figure 13. The height of the triangle can be obtained using Pythagoras' theorem. It is $\sqrt{3}$.

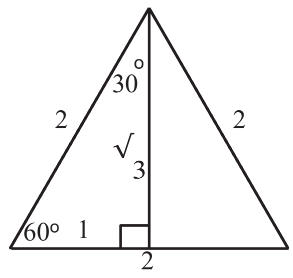


Figure 13.

From Figure 13 we can deduce

$$\sin 30^\circ = \frac{1}{2}, \qquad \cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 60^\circ = \frac{1}{2}, \qquad \tan 60^\circ = \sqrt{3}$$

All of the above results, and results for angles of 0° and 90° are summarised in the table below.

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

This is a table of sines, cosines and tangents that are all EXACT and for very specific angles: 0°, 30°, 45°, 60° and 90°. Textbooks, scientists and engineers in their everyday working and in their everyday calculations take these as being read. They assume that they are known, so you must learn them too!.

Exercise 4

In these questions you should give your answer exactly by using square roots where appropriate.

1. What is the height of an equilateral triangle with side 12 cm?
2. An isosceles triangle has base angle of 30° and base $10\sqrt{3}$ cm long. What is the height of this triangle? What is the length of the two equal sides?
3. In a right angled triangle, one angle is 45° and the side next to this angle (not the hypotenuse) has length 5 cm. What is the length of the hypotenuse?

Answers

Exercise 1
1. 5.3 m
2. 10.2 m
3. 29.7 m

Exercise 2
1. 66°
2. 42°, 22.4 m
3. 23°

Exercise 3
1. 9.1 cm, 8.5 cm
2. 6.5 cm
3. 4.6 m

Exercise 4
1. $6\sqrt{3}$ cm
2. 5 cm, 10 cm
3. $5\sqrt{2}$ cm

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Triangle formulae

mc-TY-triangleformulae-2009-1

A common mathematical problem is to find the angles or lengths of the sides of a triangle when some, but not all of these quantities are known. It is also useful to be able to calculate the area of a triangle from some of this information. In this unit we will illustrate several formulae for doing this.

In order to master the techniques explained here it is vital that you undertake the practice exercises provided.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- solve triangles using the cosine formulae
- solve triangles using the sine formulae
- find areas of triangles

Contents

1. Introduction

Consider a triangle such as that shown in Figure 1.

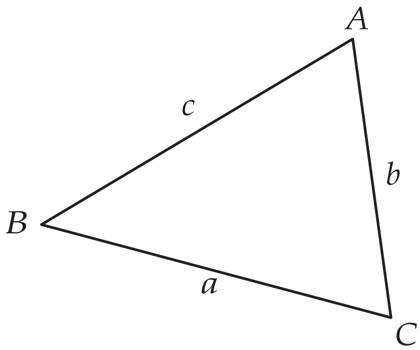


Figure 1. A triangle with six pieces of information: angles at A, B, and C; sides a, b and c.

There are six pieces of information available: angles at A, B and C, and the sides a, b and c. The angle at A is usually written A, and so on. Notice that we label the sides according to the following convention:

side b is opposite the angle B

side c is opposite the angle C

side a is opposite the angle A

Now if we take three of these six pieces of information we will (except in two special cases) be able to draw a unique triangle.

Let's deal first with the special cases.

<u>The first special case</u>

The first special case is when we know just the three angles. Then having drawn one triangle with these angles, we can draw as many more triangles as we wish, all with the same shape as the original, but larger or smaller. All will have the same angles but the sizes of the triangles will be different. We cannot define a unique triangle when we know just the three angles. This behaviour is illustrated in Figure 2 where the corresponding angles in the two triangles are the same, but clearly the triangles are of different sizes.

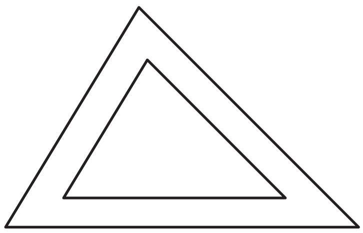


Figure 2. Given just the three angles we cannot construct a unique triangle.

The second special case

There is a second special case whereby if we are given three pieces of information it is impossible to construct a unique triangle. Suppose we are given one angle, A say, and the lengths of two of the sides. This situation is illustrated in Figure 3 (a). The first given side is marked //. The second given side is marked /; this can be placed in two different locations as shown in Figures 3b) and 3c). Consequently it is impossible to construct a unique triangle.

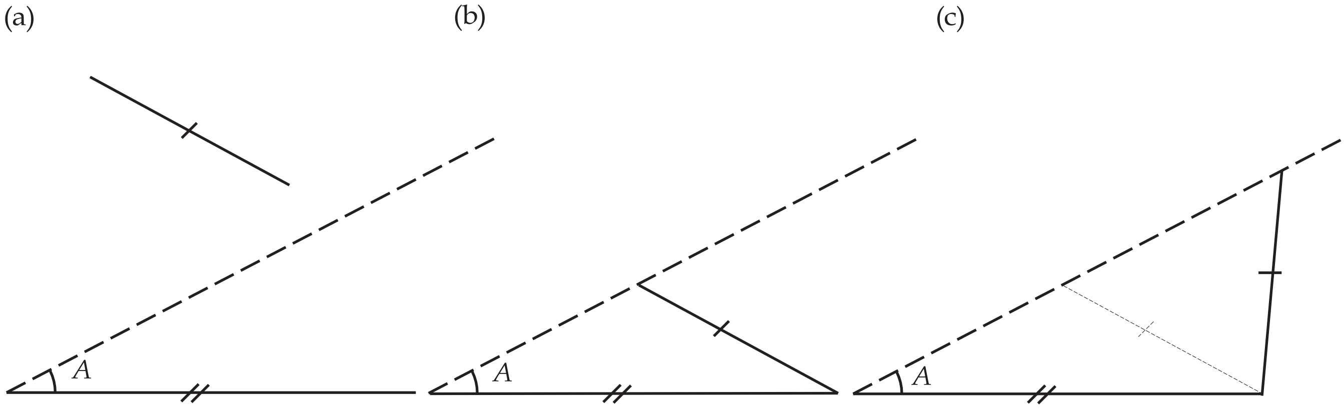


Figure 3. It is impossible to draw a unique triangle given one angle and two side lengths.

Apart from these two special cases, if we are given three pieces of information about the triangle we will be able to draw it uniquely. There are formulae for doing this which we describe in the following sections.

2. The cosine formulae

We can use the cosine formulae when three sides of the triangle are given.

Key Point

Cosine formulae

When given three sides, we can find angles from the following formulae:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

The cosine formulae given above can be rearranged into the following forms:

Key Point

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

If we consider the formula $c^2 = a^2 + b^2 - 2ab\cos C$, and refer to Figure 4 we note that we can use it to find side c when we are given two sides (a and b) and the included angle C.

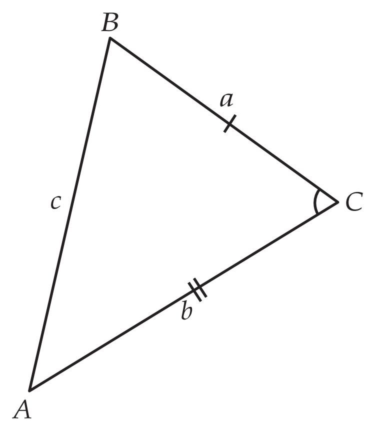


Figure 4. Using the cosine formulae to find c if we know sides a and b and the included angle C.

Similar observations can be made of the other two formulae.

So there are in fact six cosine formulae, one for each of the angles - that's three altogether, and one for each of the sides, that's another three. We only need to learn two of them, one for the angle, one for the side and then just cycle the letters through to find the others.

Exercise 1

Throughout all exercises the standard triangle notation (namely side a opposite angle A, etc.) is used.

1. Find the length of the third side, to 3 decimal places, and the other two angles, to 1 decimal place, in the following triangles

   (a) $a = 1$, $b = 2$, $C = 30^\circ$

   (b) $a = 3$, $c = 4$, $B = 50^\circ$

   (c) $b = 5$, $c = 10$, $A = 30^\circ$

2. Find the angles (to 1 decimal place) in the following triangles

   (a) $a = 2$, $b = 3$, $c = 4$

   (b) $a = 1$, $b = 1$, $c = 1.5$

   (c) $a = 2$, $b = 2$, $c = 3$

3. The sine formulae

We can use the sine formulae to find a side, given two sides and an angle which is NOT included between the two given sides.

Key Point

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the radius of the circumcircle.

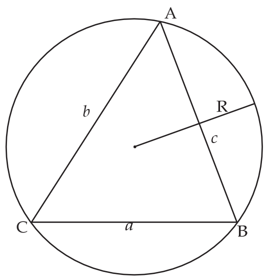


Figure 5. The circumcircle is the circle drawn through the three points of the triangle.

R is the radius of the circumcircle - the circumcircle is the circle that we can draw that will go through all the points of the triangle as shown in Figure 5.

Taking just the first three terms in the formulae we can rearrange them to give

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

and we can use the formulae in this form as well.

4. Some examples of the use of the cosine and sine formulae

Example

Suppose we are given all three sides of a triangle:

$$a = 5, \quad b = 7, \quad c = 10$$

We will use this information to determine angle A using the cosine formula:

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{7^2 + 10^2 - 5^2}{2 \times 7 \times 10} \\ &= \frac{49 + 100 - 25}{140} \\ &= \frac{124}{140} \\ A &= \cos^{-1} \frac{124}{140} = 27.7^\circ \qquad \text{(1 d.p.)} \end{aligned}$$

The remaining angles can be found by applying the other cosine formulae.

Example

Suppose we are given two sides of a triangle and an angle, as follows

$$b = 10, \quad c = 5, \quad A = 120^\circ$$

It's not immediately obvious what information we have been given. In the last Example it was very clear. So we make a sketch to mark out the information we have been given as shown in Figure 6.

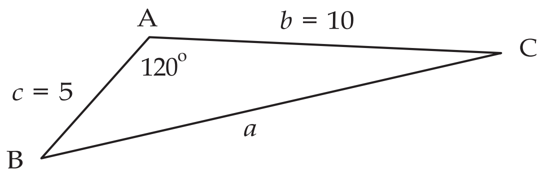


Figure 6. The information given in the Example.

From the Figure we can deduce that we have been given 2 sides and the included angle. We can use the cosine formula to deduce the length of side a.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ &= 10^2 + 5^2 - 2 \times 10 \times 5 \cos 120^\circ \\ &= 100 + 25 - 100 \cos 120^\circ \\ &= 125 - 100 \times \left(-\frac{1}{2}\right) \\ &= 175 \\ a &= \sqrt{175} = 13.2 \qquad \text{(3 s.f.)} \end{aligned}$$

Now that we have worked out the length of side a, we have three sides. We could use the cosine formulae to find out either one of the remaining angles.

Example

Suppose we are given the following information:

$$c = 8, \quad b = 12, \quad C = 30^\circ$$

Note that we are given two side lengths and an angle which is not the included angle. Referring back to the special cases described in the Introduction you will see that with this information there is the possibility that we can obtain two distinct triangles with this information.

As before we need a sketch in order to understand the information (Figure 7.)

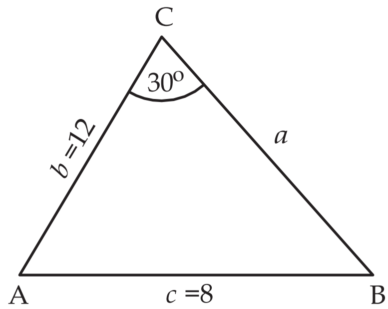


Figure 7. We are given two sides and a non-included angle.

Because we have been given two sides and a non-included angle we use the sine formulae.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Because we are given b, c and C we use the following part of the formula in order to find angle B.

$$\begin{aligned}
\frac{\sin B}{b} &= \frac{\sin C}{c} \\
\frac{\sin B}{12} &= \frac{\sin 30^\circ}{8} \\
\sin B &= \frac{12 \times \sin 30^\circ}{8} \\
&= \frac{12 \times \frac{1}{2}}{8} \\
&= \frac{6}{8} \\
&= \frac{3}{4} \\
&= 0.75 \\
B &= \sin^{-1} 0.75 \\
&= 48.6^\circ \qquad (1 \text{ d.p.})
\end{aligned}$$

Now there is a potential complication here because there is another angle with sine equal to 0.75. Specifically, B could equal $180^\circ - 48.6^\circ = 131.4^\circ$.

In the first case the angles of the triangle are then:

$C = 30^\circ$, $B = 48.6^\circ$, $A = 180^\circ - 78.6^\circ = 101.4^\circ$

In the second case we have:

$C = 30^\circ$, $B = 131.4^\circ$, $A = 180^\circ - 161.4^\circ = 18.6^\circ$.

The situation is depicted in Figure 8. In order to solve the triangle completely we must deal with the two cases separately in order to find the remaining unknown a.

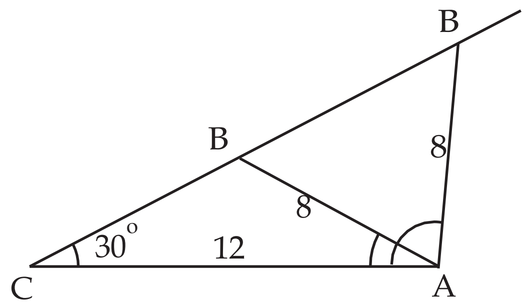


Figure 8. There are two possible triangles.

Case 1. Here $C = 30^\circ$, $B = 48.6^\circ$, $A = 101.4^\circ$. We use the sine rule in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

from which

$$\begin{aligned} a &= \frac{12 \sin 101.4^\circ}{\sin 48.6^\circ} \\ &= 15.7 \quad \text{(1 d.p.)} \end{aligned}$$

Case 2. Here $C = 30^\circ$, $B = 131.4^\circ$, $A = 18.6^\circ$. Again we can use the sine rule in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

from which

$$\begin{aligned} a &= \frac{12 \sin 18.6^\circ}{\sin 131.4^\circ} \\ &= 5.1 \quad \text{(1 d.p.)} \end{aligned}$$

Exercise 2

1. Find the lengths of the other two sides (to 3 decimal places) of the triangles with

   (a) $a = 2$, $A = 30^\circ$, $B = 40^\circ$

   (b) $b = 5$, $B = 45^\circ$, $C = 60^\circ$

   (c) $c = 3$, $A = 37^\circ$, $B = 54^\circ$

2. Find all possible triangles (give the sides to 3 decimal places and the angles to 1 decimal place) with
    (a) $a = 3$, $b = 5$, $A = 32^\circ$
    (b) $b = 2$, $c = 4$, $C = 63^\circ$
    (c) $c = 2$, $a = 1$, $B = 108^\circ$

5. The area of a triangle

We now look at a set of formulae which will give us the area of a triangle. A standard formula is

$$\text{area} = \frac{1}{2} \times \text{base} \times \text{height}$$

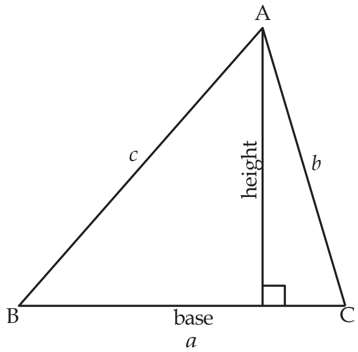


Figure 9. The area of the triangle is $\frac{1}{2}$ × base × height

Let us assume we know the lengths a, b and c, and the angle at B. Consider the right-angled triangle on the left-hand side of Figure 9. In this triangle

$$\sin B = \frac{\text{height}}{c}$$

and so, by rearranging,

$$\text{height} = c \sin B$$

Then from the formula for the area of the large triangle, $\triangle ABC$,

$$\begin{aligned} \text{area} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} ac \sin B \end{aligned}$$

Now consider the right-angled triangle on the right-hand side in Figure 9.

$$\sin C = \frac{\text{height}}{b}$$

and so, by rearranging,

$$\text{height} = b \sin C$$

So, the area of the large triangle, $\triangle ABC$, is also given by

$$\text{area} = \frac{1}{2} ab \sin C$$

It is also possible to show that the formula

$$\text{area} = \frac{1}{2} bc \sin A$$

will also give the area of the large triangle.

Key Point

When we are given two sides and the included angle, the area of the triangle can be found from one of the three formulae:

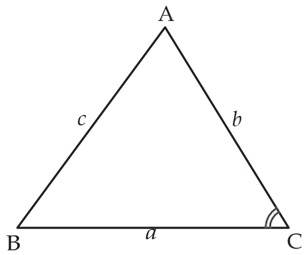


Figure 10.

$$
\begin{aligned}
\text{area} &= \frac{1}{2}ab\ \sin C \\
&= \frac{1}{2}bc\ \sin A \\
&= \frac{1}{2}ca\ \sin B
\end{aligned}
$$

These formulae do not work if we are not given an angle. An ancient Greek by the name of Hero (or Heron) derived a formula for calculating the area of a triangle when we know all three sides.

Key Point

Hero's formula:

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{where} \quad s = \frac{a+b+c}{2} = \text{semi-perimeter}$$

The semi-perimeter, as the name implies, is half of the perimeter of the triangle.

Example

Suppose we are given the lengths of three sides of a triangle:

$$a = 5 \quad b = 7 \quad c = 10$$

We can use Hero's formula:

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where

$$\begin{aligned} s &= \frac{a+b+c}{2} \\ &= \frac{5+7+10}{2} \\ &= 11 \end{aligned}$$

Then

$$\begin{aligned} \text{area} &= \sqrt{11(11-5)(11-7)(11-10)} \\ &= \sqrt{11 \times 6 \times 4 \times 1} \\ &= \sqrt{264} \\ &= 16.2 \quad (3 \text{ s.f.}) \end{aligned}$$

So the area is 16.2 square units.

Example

Suppose we wish to find the area of a triangle given the following information:

$$b = 10 \quad c = 5 \quad A = 120^\circ$$

A sketch illustrates this information.

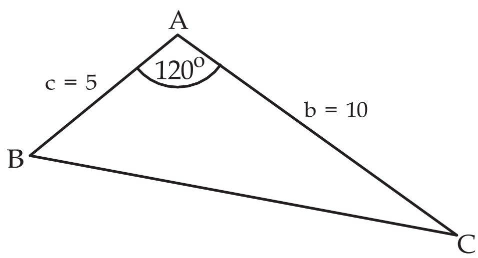


Figure 11. We are given two sides and the included angle.

We are given two sides and the included angle.

$$\begin{aligned} \text{area} &= \frac{1}{2} bc \sin A \\ &= \frac{1}{2} \times 10 \times 5 \times \sin 120 \\ &= 25 \sin 120 \\ &= 21.7 \quad (3 \text{ s.f.}) \end{aligned}$$

So the area is 21.7 square units.

Exercise 3

1. Find the areas of each of the triangles (to 3 decimal places) in Exercise 1, Question 1.
2. Find the areas of each of the triangles (to 3 decimal places) in Exercise 1, Question 2.

6. Summary

Cosine Formulae

for finding an angle using the three sides:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

for finding a side using two sides and the included angle

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = c^2 + a^2 - 2ca \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Sine Formulae

Use when you are given two sides and the non-included angle, or two angles and a side:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Formulae for the area of a triangle

$$\text{area} = \frac{1}{2}ab \sin C$$

$$= \frac{1}{2}bc \sin A$$

$$= \frac{1}{2}ca \sin B$$

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where

$$s = \frac{a+b+c}{2} = \text{semi-perimeter}$$

Exercise 4

1. Determine the lengths of all the sides (to 3 decimal places), the sizes of all the angles (to 1 decimal place) and the area (to 3 decimal places) of each of the following triangles.

   (a) $a = 5$, $b = 3$, $c = 6$
   (b) $a = 5$, $b = 2$, $C = 42^\circ$
   (c) $c = 3$, $A = 40^\circ$, $B = 60^\circ$
   (d) $b = 2$, $A = 73^\circ$, $B = 41^\circ$
   (e) $a = 5$, $b = 3$, $c = 4$
   (f) $c = 2$, $b = 5$, $B = 78^\circ$

Answers

Exercise 1
1. (a) $c = 1.239$, $A = 23.8^\circ$, $B = 126.2^\circ$ (b) $b = 3.094$, $A = 48.0^\circ$, $C = 82.0^\circ$
(c) $a = 6.197$, $B = 23.8^\circ$, $C = 126.2^\circ$.

2. (a) $A = 29.0^\circ$, $B = 46.6^\circ$, $C = 104.5^\circ$ (b) $A = 41.4^\circ$, $B = 41.4^\circ$, $C = 97.2^\circ$
(c) $A = 41.4^\circ$, $B = 41.4^\circ$, $C = 97.2^\circ$

Exercise 2
1. (a) $b = 2.571$, $c = 3.759$ (b) $a = 7.044$, $c = 6.124$ (c) $a = 1.806$, $b = 2.427$

2. (a) Two possible triangles:
$B = 62.0^\circ$, $C = 86.0^\circ$, $c = 5.647$ and $B = 118.0^\circ$, $C = 30.0^\circ$, $c = 2.833$
(b) $B = 27.0^\circ$, $A = 88.0^\circ$, $a = 4.487$ (c) $b = 2.457$, $A = 23.2^\circ$, $C = 51.9^\circ$

Exercise 3
1. (a) 0.5 (b) 0.587 (c) 12.5
2. (a) 2.905 (b) 0.496 (c) 1.984

Exercise 4
1. (a) $A = 56.3^\circ$, $B = 29.9^\circ$, $C = 93.8^\circ$, Area = 7.483
(b) $c = 3.760$, $A = 117.2^\circ$, $B = 20.9^\circ$, Area = 3.346
(c) $a = 1.958$, $b = 2.638$, $C = 80^\circ$, Area = 2.544
(d) $a = 2.915$, $c = 2.785$, $C = 66^\circ$, Area = 2.663
(e) $A = 90^\circ$, $B = 36.9^\circ$, $C = 53.1^\circ$, Area = 6
(f) $a = 5.017$, $A = 79.0^\circ$, $C = 23.0^\circ$, Area = 4.908

Trigonometric ratios of an angle of any size

mc-TY-trigratiosanysize-2009-1

Knowledge of the trigonometrical ratios sine, cosine and tangent, is vital in very many fields of engineering, mathematics and physics. This unit explains how the sine, cosine and tangent of an arbitrarily sized angle can be found.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- define the ratios sine, cosine and tangent with reference to projections.
- use the trig ratios to solve problems involving triangles.

Contents

1. Introduction

In this session we are going to be looking at the definitions of sine, cosine and tangent for any size of angle. Let's first of all recall sine, cosine and tangent for angles in a right-angled triangle.

2. Trig ratios for angles in a right-angled triangle

Refer to the triangle in Figure 1.

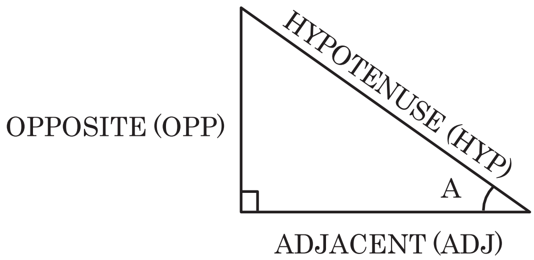


Figure 1. The side opposite the right-angle is called the hypotenuse

The side that is the longest side in the right-angled triangle and that is opposite the right angle is called the hypotenuse, or HYP for short.

The side that is opposite the angle A is called the opposite side, or OPP for short.

The side that runs alongside the angle A, and which is not the hypotenuse is called the adjacent side, or ADJ for short.

Recall the following important definitions:

Key Point

$$\sin A = \frac{\text{OPP}}{\text{HYP}}$$

$$\cos A = \frac{\text{ADJ}}{\text{HYP}}$$

$$\tan A = \frac{\text{OPP}}{\text{ADJ}}$$

However, these are defined only for acute angles, these are angles less than 90°. What happens if we have an angle greater than 90°, or less than 0° ? We explore this in the following section.

3. Angles

Consider Figure 2 which shows a circle of radius 1 unit centred at the origin. Imagine a point P on the circle which moves around the circle in an anticlockwise sense.

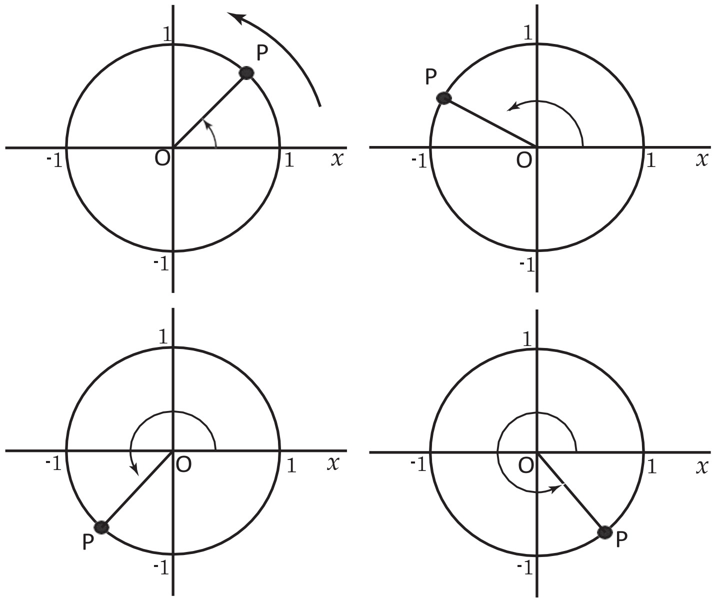


Figure 2. Angles measured anticlockwise from the positive x axis are deemed to be positive angles.

In the first diagram in Figure 2 the angle is acute, that is, it is greater than 0° but less than 90°. When P moves into the second quadrant, the angle lies between 90° and 180°. The angle is now obtuse. When P moves into the third quadrant, the angle is greater than 180° but less than 270°. Finally in the fourth quadrant, the angle is greater than 270° but less that 360°.

Consider now Figure 3. On these diagrams the arm OP is moving in a clockwise sense from the positive x axis. Such angles are conventionally taken to be negative angles.

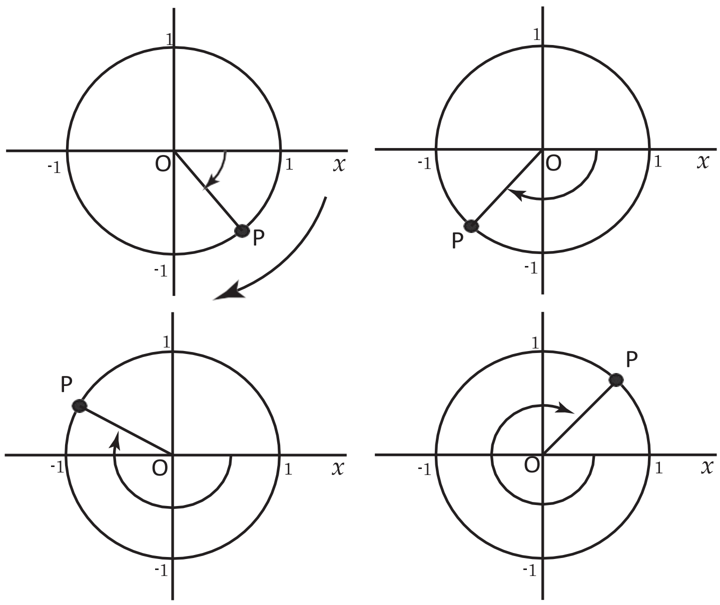


Figure 3. Angles measured clockwise from the positive x axis are deemed to be negative angles.

So, in this way we understand what is meant by an angle of any size, positive or negative. We now use these ideas together with our earlier definitions of sine, cosine and tangent in order to define these trig ratios for angles of any size.

4. The sine of an angle in any quadrant

Consider Figure 4 which shows a circle of radius 1 unit.

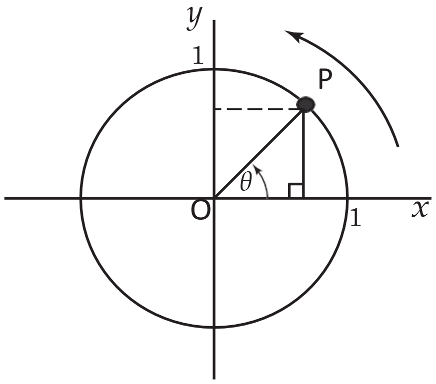


Figure 4. The side opposite θ has the same length as the projection of OP onto the y axis Oy.

The arm OP is in the first quadrant and we have dropped a perpendicular line down from P to the x axis in order to form the right-angled triangle shown.

Consider angle θ. The side opposite this angle has the same length as the projection of OP onto the y axis. So we define

$$\sin\theta = \frac{\text{projection of } OP \text{ onto } Oy}{OP}$$

$$= \text{projection of } OP \text{ onto } Oy$$

since OP has length 1. This is entirely consistent with our earlier definition of $\sin\theta$ as $\frac{\text{OPP}}{\text{HYP}}$.

Moreover, we can use this new definition to find the sine of any angle. Note that when the arm OP has rotated into the third and fourth quadrants the projection onto Oy will be negative.

Let's have a look at what that means in terms of a graph. Figure 5 shows the unit circle and the arm in various positions. The graph alongside is the projection of the arm onto the y axis. Corresponding points on both the circle and the graph are labelled A, B, C and so on. In other words this is the graph of $\sin\theta$.

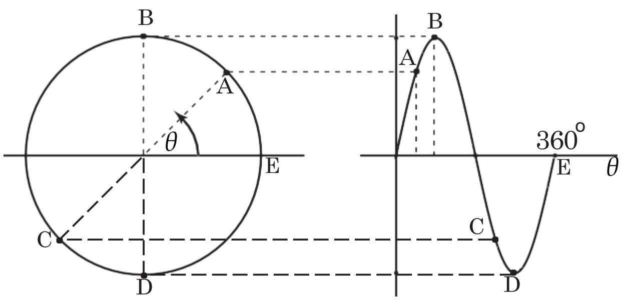


Figure 5. The graph of $\sin\theta$ can be drawn from the projections of the arm onto the y axis

We can produce a similar diagram for negative angles and we will obtain the graph shown in Figure 6. The whole pattern is reproduced every 360°. In this way we can find the sine of any angle at all. Note also that the sine has a maximum value of 1, and a minimum value of -1. The graph never moves outside this range of values. To the left of -360° and to the right of

$+360^\circ$ the basic pattern simply repeats. This behaviour corresponds to arm OP moving around the circle again.

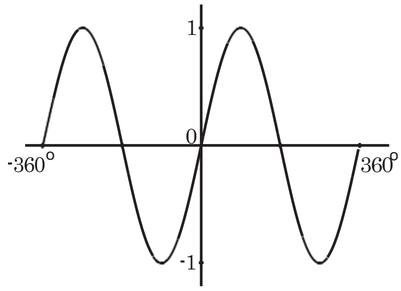


Figure 6. The graph of $\sin\theta$ extended to include negative angles

5. The cosine of any angle

To explore the cosine graph refer to Figure 7.

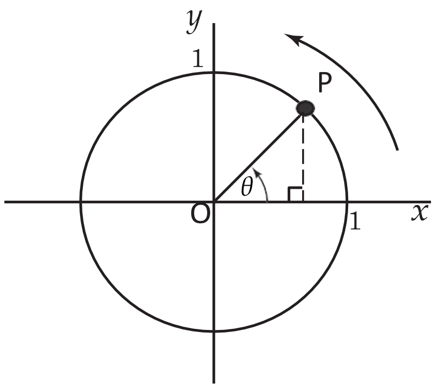


Figure 7. The side adjacent to angle θ has a length equal to the projection of OP onto the x axis.

We know that $\cos\theta = \frac{\text{ADJ}}{\text{HYP}}$. The length of the adjacent side is the same as the length of the projection of the arm OP onto the x axis. Hence we take the following as our definition of cosine:

$$\cos\theta = \frac{\text{projection of } OP \text{ onto } Ox}{OP}$$

$$= \text{projection of } OP \text{ onto } Ox$$

since we are considering a unit circle and so $OP = 1$.

We can produce a graph as we did previously for $\sin\theta$ by finding the length of the projection of the arm OP onto the x axis. This is done by looking down on the arm from above as shown in Figure 8. For example, when $\theta = 0$, (point A), the projection has length 1. When $\theta = 90^\circ$, the projection looks like a single point and has length zero (point B). When θ moves into the second and third quadrants, the x projection, and hence $\cos\theta$, is negative. In the fourth quadrant, the

x projection, and hence $\cos\theta$, is positive.

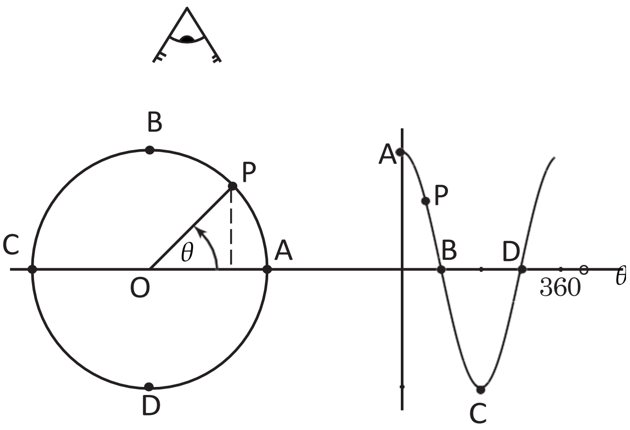


Figure 8. Look down on OP from above to find the projection of OP onto the x axis.

We can continue in this fashion to produce a cosine graph for negative angles. Doing so will result in the graph shown in Figure 9.

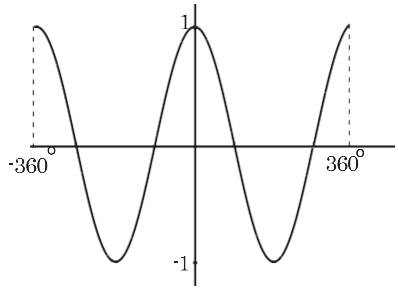


Figure 9. The graph of $\cos\theta$ extended to include negative angles.

This is a periodic graph. The same shape repeats every 360° as we move further to the left and to the right. Note also that the cosine has a maximum value of 1, and a minimum value of -1. The graph never moves outside this range of values.

Another important point to note is that the sine and cosine curves have the same shape. The cosine graph is the same as the sine except that it is displaced by 90°.

6. The tangent of any angle

Recall that tangent has already been defined as $\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}}$.

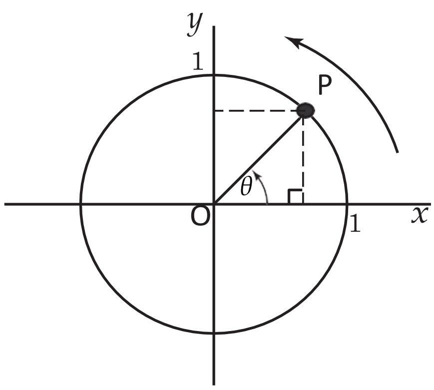


Figure 10. To find $\tan\theta$ we need to use projections onto both axes.

In terms of projections this definition becomes:

$$\tan\theta = \frac{\text{projection of } OP \text{ onto } Oy}{\text{projection of } OP \text{ onto } Ox}$$

An important result which follows immediately from comparing this definition with the earlier ones for sine and cosine is that:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

This gives us an identity which we need to learn and remember.

What does the graph of tangent look like? It's a little bit trickier to draw, but can be done by considering projections as outlined in Figure 11.

For example at point A, the projection onto Oy is zero, whilst the projection onto Ox is 1. Hence at A, $\tan\theta = \frac{0}{1} = 0$ and the corresponding point is indicated on the graph.

At point B both projections are equal and so $\tan\theta = 1$.

At points near to C the projection onto Oy is approaching 1, whilst the projection onto Ox is approaching zero. Hence the ratio

$$\tan\theta = \frac{\text{projection of } OP \text{ onto } Oy}{\text{projection of } OP \text{ onto } Ox}$$

becomes very large indeed. We indicate this by the dotted line on the graph. This line, called an asymptote, is approached by the graph as θ approaches 90°.

Continuing in this fashion we can produce the graph of $\tan\theta$ for any angle θ, as shown in Figure 11.

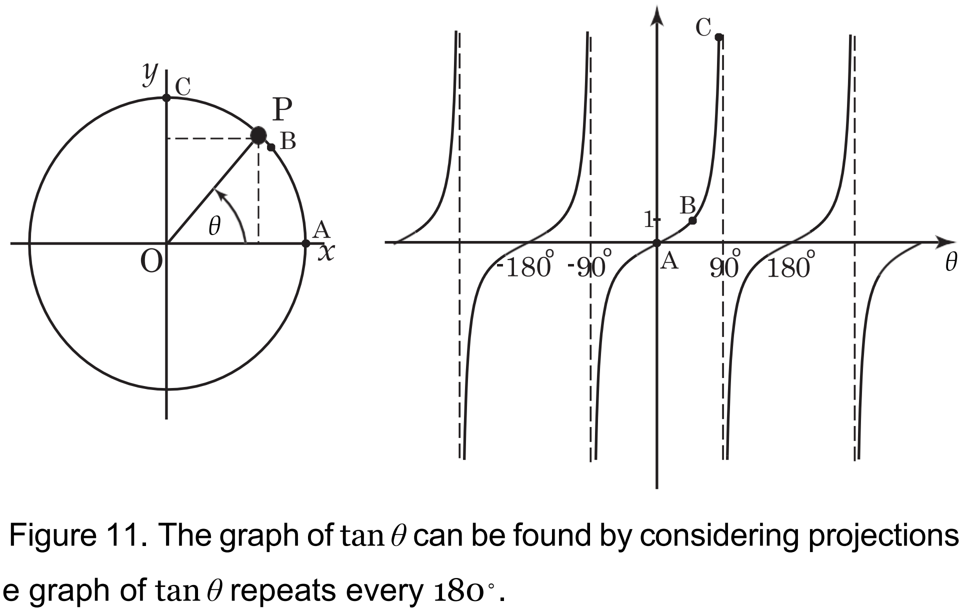


Figure 11. The graph of $\tan\theta$ can be found by considering projections

Note that the graph of $\tan\theta$ repeats every 180°.

Exercise 1

Determine whether each of the following statements is true or false.

1. Sine is positive in the 1st and 4th quadrants.

2. The graph of cosine repeats itself every 180^o
3. The graph of tangent repeats itself every 180^o
4. Cosine is negative in the 2nd and 3rd quadrants
5. The graph of sine is continuous (i.e. has no breaks)
6. Tangent is negative in the 2nd and 4th quadrants
7. The graph of tangent is continuous (i.e. has no breaks)

Answers

Exercise 1

1. False 2. False 3. True 4. True 5. True 6. True 7. False

Radians

mc-TY-radians-2009-1

At school we usually learn to measure an angle in degrees. However, there are other ways of measuring an angle. One that we are going to have a look at here is measuring angles in units called radians. In many scientific and engineering calculations radians are used in preference to degrees.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- use radians to measure angles
- convert angles in radians to angles in degrees and vice versa
- find the length of an arc of a circle
- find the area of a sector of a circle
- find the area of a segment of a circle

Contents

1. Introduction

At school we usually learn to measure an angle in degrees. We are well aware that a full rotation is 360° as shown in Figure 1.

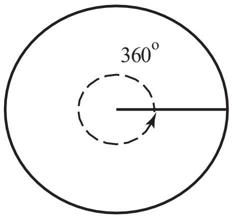


Figure 1. A full rotation is 360°.

However, there are other ways of measuring an angle. One way that we are going to have a look at here is measuring angles in units called radians. In many scientific and engineering calculations radians are used in preference to degrees.

2. Definition of a radian

Consider a circle of radius r as shown in Figure 2.

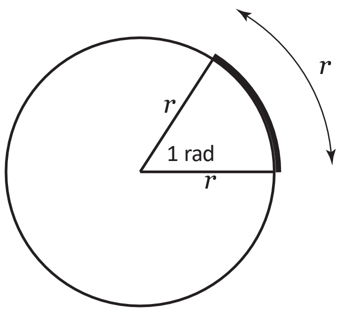


Figure 2. The arc shown has a length chosen to equal the radius; the angle is then 1 radian.

In Figure 2 we have highlighted part of the circumference of the circle chosen to have the same length as the radius. The angle at the centre, so formed, is 1 radian.

Key Point

An angle of one radian is subtended by an arc having the same length as the radius as shown in Figure 2.

3. Arc length

We will now use this definition to find a formula for the length of an arbitrary arc.

We have seen that an angle of 1 radian is subtended by an arc of length r as illustrated in the left-most diagram in Figure 3. By extension an angle of 2 radians will be subtended by an arc of length $2r$, as shown.

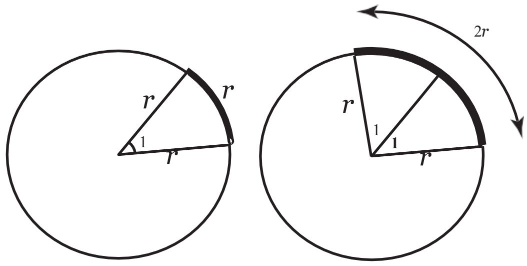


Figure 3. An angle of 2 radians is subtended by an arc of length $2r$.

Note from these diagrams that the length of the arc is always given by

the angle in radians × the radius

In the general case, the length s, of an arbitrary arc which subtends an angle θ is $r\theta$ as illustrated in Figure 4.

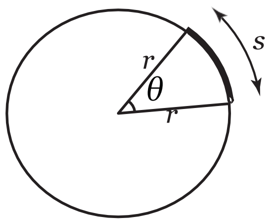


Figure 4. The arc length s, is given by $r \times \theta$

This gives us a way of calculating the arc length when we know the angle at the centre of the circle and we know its radius.

Key Point

arc length $s = r\theta$

(note: θ must be measured in radians)

Exercise 1

Determine the angle (in radians) subtended at the centre of a circle of radius 3cm by each of the following arcs:

a) arc of length 6 cm b) arc of length 3π cm

c) arc of length 1.5 cm d) arc of length 6π cm

4. Equivalent angles in degrees and in radians

We know that the arc length for a full circle is the same as its circumference, $2\pi r$.

We also know that the arc length = $r\theta$.

So for a full circle

$$2\pi r = r\theta$$

that is

$$\theta = 2\pi$$

In other words, when we are working in radians, the angle in a full circle is 2π radians, in other words

$$360^\circ = 2\pi \text{ radians}$$

This enables us to have a set of equivalences between degrees and radians.

Key Point

$$360^\circ = 2\pi \text{ radians}$$

from which it follows that

$$180^\circ = \pi \text{ radians} \qquad 90^\circ = \frac{\pi}{2} \text{ radians}$$

$$45^\circ = \frac{\pi}{4} \text{ radians} \qquad 60^\circ = \frac{\pi}{3} \text{ radians}$$

$$30^\circ = \frac{\pi}{6} \text{ radians}$$

The Key Point gives a list of angles measured in degrees on the left and the equivalent list in radians on the right. It is important in mathematical work that you record correctly the unit of measure you are using.

Another useful relationship is given as follows:

$$\pi \text{ radians} = 180^\circ$$

so

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} = 57.296^\circ \qquad \text{(3 d.p.)}$$

So 1 radian is just over 57°.

Some notation.

There are various conventions used to denote radians. Some books and some teachers use ‘rads’ as in 2 rads. Others use a small c as in 2^c. Some others use no symbol at all and assume that radians are being used. When an angle is expressed as a multiple of π, for example as in the expression $\sin\frac{3\pi}{2}$, it is taken as read that the angle is being measured in radians.

Exercise 2

1. When each of the following angles is converted from degrees to radians the answer can be expressed as a multiple of π (note that it may be a fractional multiple). In each case state the multiple (e.g for an answer of $\frac{4\pi}{5}$ the multiple is $\frac{4}{5}$).
   a) 90^o b) 360^o c) 60^o d) 45^o
   e) 120^o f) 15^o g) 135^o h) 270^o

2. Convert each of the following angles from radians to degrees.
   a) $\frac{\pi}{2}$ radians b) $\frac{3\pi}{4}$ radians c) π radians d) $\frac{\pi}{6}$ radians
   e) 5π radians f) $\frac{4\pi}{5}$ radians g) $\frac{7\pi}{4}$ radians h) $\frac{\pi}{4}$ radians

3. Convert each of the following angles from degrees to radians giving your answer to 2 decimal places.
   a) 17^o b) 49^o c) 124^o d) 200^o

4. Convert each of the following angles from radians to degrees, giving your answer to 1 decimal place.
   a) 0.6 radians b) 2.1 radians c) 3.14 radians d) 1 radian

5. Finding an arc length when the angle is given in degrees

We know that if θ is measured in radians, then the length of an arc is given by $s = r\theta$.

Suppose θ is measured in degrees. We shall derive a new formula for the arc length.

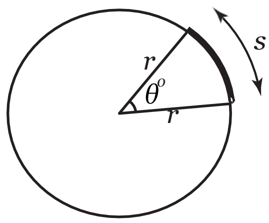


Figure 5. In this circle the angle θ is measured in degrees.

Referring to Figure 5, the ratio of the arc length to the full circumference will be the same as the ratio of the angle subtended by the arc, to the angle in a full circle; that is

$$\frac{s}{2\pi r} = \frac{\theta^\circ}{360^\circ}$$

So, when θ is measured in degrees we can use the following formula for arc length:

$$s = 2\pi r \times \frac{\theta^\circ}{360^\circ}$$

Notice how the earlier formula, used when the angle is measured in radians, is much simpler.

6. The area of a sector of a circle

A sector of a circle with angle θ is shown shaded in Figure 6.

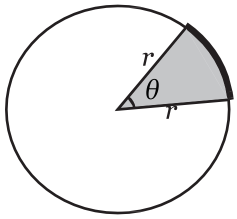


Figure 6. The shaded area is a sector of the circle.

The ratio of the area of the sector to the area of the full circle will be the same as the ratio of the angle θ to the angle in a full circle. The full circle has area πr^2. Therefore

$$\frac{\text{area of sector}}{\text{area of full circle}} = \frac{\theta}{2\pi}$$

and so

$$\begin{aligned}\text{area of sector} &= \frac{\theta}{2\pi} \times \pi r^2 \\ &= \frac{1}{2} r^2 \theta\end{aligned}$$

Key Point

$$\text{area of sector} = \frac{1}{2} r^2 \theta$$

when θ is measured in radians

7. Miscellaneous Examples

Example

Consider the circle shown in Figure 7. Suppose we wish to calculate the angle θ.

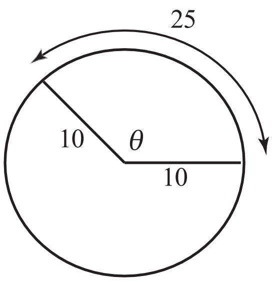


Figure 7. Calculate the angle θ.

We know the arc length and radius. We can use the formula $s = r\theta$. Substituting the given values

$$25 = 10\theta$$

and so

$$\theta = \frac{25}{10} = 2.5 \text{ rads}$$

What is this angle in degrees ? We know

$$\pi \text{ rads} = 180^\circ$$

and so

$$1 \text{ rad} = \frac{180^\circ}{\pi}$$

It follows that

$$2.5 \text{ rads} = 2.5 \times \frac{180^\circ}{\pi} = 143.2^\circ$$

Example

Refer to Figure 8. Suppose we have a circle of radius 10cm and an arc of length 15cm. Suppose we want to find (a) the angle θ, (b) the area of the sector OAB, (c) the area of the minor segment (shaded).

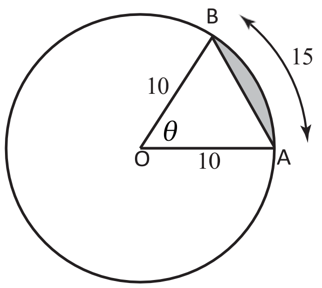


Figure 8. The shaded area is called the minor segment.

(a) Using $s = r\theta$ we have $15 = 10\theta$ and so $\theta = \frac{15}{10} = 1.5^c$.

(b) Using the formula for the area of the sector, $A = \frac{1}{2}r^2\theta$, we find

$$\begin{aligned} \text{area} &= \frac{1}{2}r^2\theta \\ &= \frac{1}{2}(10^2)(1.5) \\ &= 75 \text{ cm}^2 \end{aligned}$$

(c) We already know that the area of the sector OAB is 75cm^2. If we can work out the area of the triangle AOB we can then determine the area of the minor segment. (Recall the formulae for the area of triangle, $A = \frac{1}{2}ab\sin C$.)

$$\begin{aligned} \text{area of triangle} &= \frac{1}{2}r^2 \sin\theta \\ &= \frac{1}{2}10^2 \sin 1.5 \\ &= 49.87 \text{ cm}^2 \end{aligned}$$

Therefore the area of the minor segment is

$$75 - 49.87 = 25.13 \text{ cm}^2 \qquad \text{(to 2 dp.)}$$

Example

Suppose we have an angle of 120°. What is this angle in radians ? We know that

$$\pi \text{ rads} = 180^\circ$$

and so

$$\frac{\pi}{180} \text{ rads} = 1^\circ$$

then

$$120^\circ = \frac{\pi}{180} \times 120 \text{ rads}$$

This can be written as $\frac{2\pi}{3}$ radians (= 2.094 radians).

Exercise 3

A sector of a circle is an area bounded by two radii and an arc. A sector has an angle at the centre of the circle. All the questions below relate to a circle with radius 5cm.

1. Determine the length of the arc (correct to 2 decimal places) when the angle at the centre is a) 1.2 radians b) $\frac{\pi}{2}$ radians c) 45°
2. Calculate the area (correct to 2 decimal places) of each of the three sectors in Question 1.
3. A sector of this circle has area 50 cm^2. What is the angle (in radians) at the centre of this sector?

Answers

Exercise 1

a) 2 b) π c) 0.5 d) 2π

Exercise 2

1. a) $\frac{1}{2}$ b) 2 c) $\frac{1}{3}$ d) $\frac{1}{4}$ e) $\frac{2}{3}$ f) $\frac{1}{12}$ g) $\frac{3}{4}$ h) $\frac{3}{2}$
2. a) 90° b) 135° c) 180° d) 30° e) 900° f) 144° g) 315° h) 18°
3. a) 0.30 radians b) 0.86 radians c) 2.16 radians d) 3.49 radians
4. a) 15.3° b) 120.3° c) 179.9° d) 57.3°

Exercise 3

1. a) 6 cm b) 7.85 cm c) 3.93 cm
2. a) 15 cm^2 b) 19.63 cm^2 c) 9.82 cm^2
3. 4 radians

Introduction to vectors

mc-TY-introvector-2009-1

A vector is a quantity that has both a magnitude (or size) and a direction. Both of these properties must be given in order to specify a vector completely. In this unit we describe how to write down vectors, how to add and subtract them, and how to use them in geometry.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

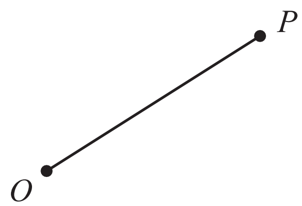
After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- distinguish between a vector and a scalar;
- understand how to add and subtract vectors;
- know when one vector is a multiple of another;
- use vectors to solve simple problems in geometry.

1. Introduction

Vector quantities are extremely useful in physics. The important characteristic of a vector quantity is that it has both a magnitude (or size) and a direction. Both of these properties must be given in order to specify a vector completely.

An example of a vector quantity is a displacement. This tell us how far away we are from a fixed point, and it also tells us our direction relative to that point.



Another example of a vector quantity is velocity. This is speed, in a particular direction. An example of velocity might be 60 mph due north.

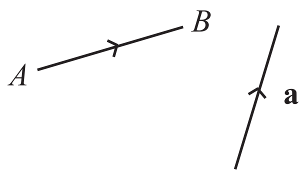
A quantity with magnitude alone, but no direction, is not a vector. It is called a *scalar* instead. One example of a scalar is distance. This tells us how far we are from a fixed point, but does not give us any information about the direction. Another example of a scalar quantity is the mass of an object.

Key Point

A vector has both magnitude and direction, and both these properties must be given in order to specify it. A quantity with magnitude but no direction is called a scalar.

2. Representing vector quantities

We can represent a vector by a line segment. This diagram shows two vectors.

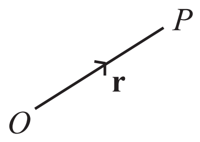


We have used a small arrow to indicate that the first vector is pointing from A to B. A vector pointing from B to A would be going in the opposite direction.

Sometimes we represent a vector with a small letter such as $\mathbf{a}$, in a bold typeface. This is common in textbooks, but it is inconvenient in handwriting. In writing, we normally put a bar underneath, or sometimes on top of, the letter: $\underline{a}$ or $\bar{a}$. In speech, we call this the vector "a-bar".

3. Position vectors

Sometimes vectors are referred to a fixed point, an origin. Such a vector is called a position vector. So we might refer to the position vector of a point P with respect to an origin O. In writing, might put $\overline{OP}$ for this vector. Alternatively, we could write it as r. These two expressions refer to the same vector.



4. Some notation for vectors

What does it mean if, for two vectors, $\mathrm{a} = \mathrm{b}$? This means first that the length of a equals the length of b, so that the two vectors have the same magnitude. But it also means that a and b are in the same direction. How can we write this down more succinctly?

If two vectors are "in the same direction", then they are parallel. We write this down as $\mathrm{a} /\!/ \mathrm{b}$.

For length, if we have a vector $\overline{AB}$, we can write its length as AB without the bar. Alternatively, we can write it as $|\overline{AB}|$. The two vertical lines give us the modulus, or size of, the vector. If we have a vector written as a, we can write its length as either $|\mathrm{a}|$ with two vertical lines, or as a in ordinary type (or without the bar). This is why it is very important to keep to the convention that has been adopted in order to distinguish between a vector and its length.

Key Point

The length of a vector $\overline{AB}$ is written as

$$AB \text{ or } |\overline{AB}|,$$

and the length of a vector a is written as

a (in ordinary type, or without the bar) or as $|\mathrm{a}|$.

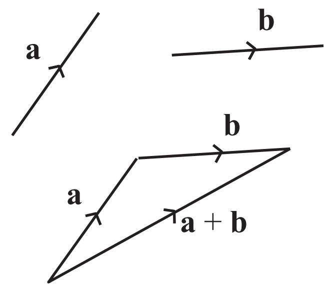
If two vectors a and b are parallel, we write

$$\mathrm{a} /\!/ \mathrm{b}$$

5. Adding two vectors

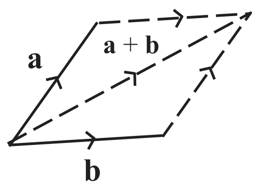
One of the things we can do with vectors is to add them together. We shall start by adding two vectors together. Once we have done that, we can add any number of vectors together by adding the first two, then adding the result to the third, and so on.

In order to add two vectors, we think of them as displacements. We carry out the first displacement, and then the second. So the second displacement must start where the first one finishes.



The sum of the vectors, $\mathbf{a} + \mathbf{b}$ (or the *resultant*, as it is sometimes called) is what we get when we join up the triangle. This is called the *triangle law* for adding vectors.

There is another way of adding two vectors. Instead of making the second vector start where the first one finishes, we make them both start at the same place, and complete a parallelogram. This is called the *parallelogram law* for adding vectors. It gives the same result as the triangle law, because one of the properties of a parallelogram is that opposite sides are equal and in the same direction, so that $\mathbf{b}$ is repeated at the top of the parallelogram.

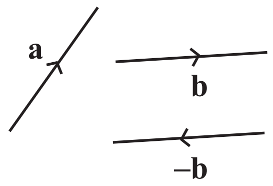


Key Point

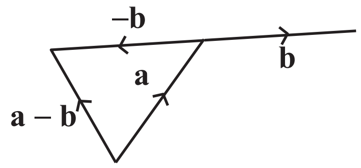
We can add two vectors $\mathbf{a}$ and $\mathbf{b}$ by making $\mathbf{b}$ start where $\mathbf{a}$ finishes, and completing the triangle. Alternatively, we can make $\mathbf{a}$ and $\mathbf{b}$ start at the same place, and take the diagonal of the parallelogram.

6. Subtracting two vectors

What is $\mathbf{a} - \mathbf{b}$? We think of this as $\mathbf{a} + (-\mathbf{b})$, and then we ask what $-\mathbf{b}$ might mean. This will be a vector equal in magnitude to $\mathbf{b}$, but in the reverse direction.



Now we can subtract two vectors. Subtracting $\mathbf{b}$ from $\mathbf{a}$ will be the same as *adding* $-\mathbf{b}$ to $\mathbf{a}$.

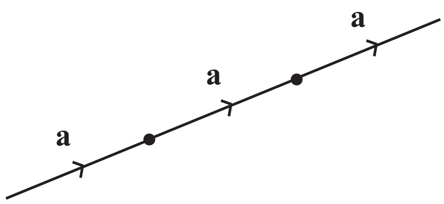


Key Point

$\mathbf{a} - \mathbf{b}$ means $\mathbf{a} + (-\mathbf{b})$

7. Adding a vector to itself

What happens when you add a vector to itself, perhaps several times? We write, for example, $\mathbf{a} + \mathbf{a} + \mathbf{a} = 3\mathbf{a}$.



In the same way, we would write

$$n\mathbf{a} = \underbrace{\mathbf{a} + \ldots + \mathbf{a}}_{n \text{ copies}}.$$

Key Point

A vector $n\mathbf{a}$ is in the same direction as the vector $\mathbf{a}$, but n times as long.

8. Vectors of unit length

There is one more piece of notation we shall use when writing vectors. If $\mathbf{a}$ is any vector, we shall write $\hat{\mathbf{a}}$ to represent a unit vector in the direction of $\mathbf{a}$. A unt vector is a vector whose length is 1, so that

$$|\hat{\mathbf{a}}| = 1.$$

This notation gives us another way of writing the vector $\mathbf{a}$: we can write it as $a\,\hat{\mathbf{a}}$, so that it is the length a multiplied by the unit vector $\hat{\mathbf{a}}$.

Key Point

A unit vector in the direction of the vector $\mathbf{a}$ is written as $\hat{\mathbf{a}}$, so that

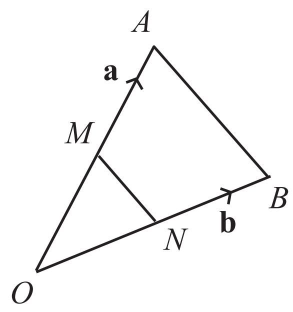
$$\mathbf{a} = a\,\hat{\mathbf{a}}.$$

9. Using vectors in geometry

Example

There is a useful theorem in geometry called the *mid-point theorem*. In this theorem, we take two points A and B, defined with respect to an origin O. Let us write $\mathbf{a}$ for the position vector of A, and $\mathbf{b}$ for the position vector of B. We can join A and B with a line, to give a triangle.

Now take the mid-point M of the line OA, and the mid-point N of the line OB, and join M to N with a line. Can we say anything about the relationship between the line MN and the line AB?



We can answer this very easily with vectors. We can write the vector for the line segment $\overline{AB}$ as $\overline{AO} + \overline{OB}$. Now $\overline{AO}$ is the reverse of the vector a, so it is −a. And $\overline{OB}$ is the same as the vector b. Therefore

$$\begin{aligned} \overline{AB} &= \overline{AO} + \overline{OB} \\ &= (-\mathrm{a}) + \mathrm{b} \\ &= \mathrm{b} - \mathrm{a}. \end{aligned}$$

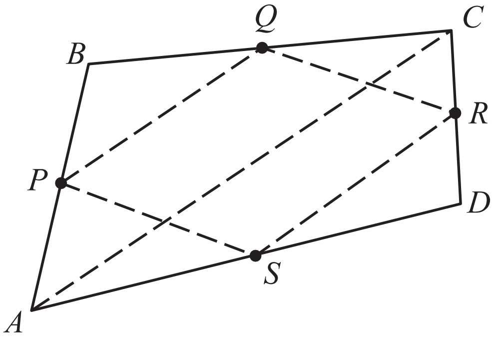
What about $\overline{MN}$? Following the same reasoning, this is $\overline{MO} + \overline{ON}$. But what is $\overline{MO}$? This is a vector half the length of $\overline{AO}$, and in the same direction, so it must be $\frac{1}{2}(-\mathrm{a})$. In the same way, $\overline{ON}$ is in the same direction as $\overline{OB}$, but is half the length, so it must be $\frac{1}{2}\mathrm{b}$. Therefore

$$\begin{aligned} \overline{MN} &= \overline{MO} + \overline{ON} \\ &= \tfrac{1}{2}(-\mathrm{a}) + \tfrac{1}{2}\mathrm{b} \\ &= \tfrac{1}{2}(\mathrm{b} - \mathrm{a}). \end{aligned}$$

Now we can compare $\overline{AB}$ and $\overline{MN}$. From our calculation, we can see that $\overline{MN}$ is $\frac{1}{2}\overline{AB}$. So, as this is a vector equation, it tells us two things. First, it tells us about magnitude, so that $MN = \frac{1}{2}AB$. Also, it tells us that MN and AB must be in the same direction, so that $MN // AB$. This is called the mid-point theorem for a triangle. It states that if you join the mid-points of two sides of a triangle then the resulting line is equal to half of the third side of the triangle, and is parallel to it.

Example

We can apply the mid-point theorem to a quadrilateral, or indeed to any four points in space, to give an interesting geometrical result. We shall call the four points A, B, C and D. We shall also give labels to the mid-points of the four sides: we shall call the mid-points P, Q, R and S. Now let us join the four mid-points, to make a new shape $PQRS$. What kind of shape is this?



We can identify the shape by joining the points A and C.
If we apply the mid-point theorem to triangle ABC, we see that

$$\overline{PQ} = \tfrac{1}{2}\overline{AC}.$$

But if we apply the mid-point theorem to the triangle ADC, we also see that

$$\overline{RS} = \tfrac{1}{2}\overline{AC}.$$

If we combine these two equations, we then obtain

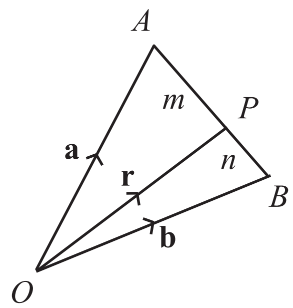
$$\overline{PQ} = \overline{RS}.$$

Now this is a vector equation, and so it tells us two things. First, it tells us that the length of PQ is the same as the length of RS. And secondly, it tells us that the direction of PQ is the same as the direction of RS, so that PQ and RS are parallel. But having two parallel sides of equal length is a property which defines a parallelogram, and so the shape $PQRS$ must be a parallelogram.

Example

We shall now use vectors to prove one more theorem.

Take two points A and B, having position vectors a, b with respect to an origin O. Draw the line AB, and take a point P on that line which divides it in the ratio of m to n. What is the position vector of P with respect to O?.



We can use the same method that we used before. We know that

$$\overline{OP} = \overline{OA} + \overline{AP}, \qquad (1)$$

and we also know that $\overline{OA} = \mathrm{a}$. But what is $\overline{AP}$?
Now $\overline{AP}$ is in the same direction as $\overline{AB}$, and their lengths are in the ratio of m to $m+n$. So

$$\overline{AP} = \frac{m}{m+n}\overline{AB}. \qquad (2)$$

We also know that

$$\begin{aligned} \overline{AB} &= \overline{AO} + \overline{OB} \\ &= \mathrm{b} - \mathrm{a}. \end{aligned}$$

Now we can put these three statements together, replacing $\overline{AP}$ in equation (1) by using equation (2), and replacing $\overline{AB}$ in equation (2) by using the equation (3), so that everything will be written in terms of $\mathbf{a}$ and $\mathbf{b}$. This gives us

$$\overline{OP} = \mathbf{a} + \frac{m}{m+n}(\mathbf{b} - \mathbf{a}).$$

Putting all this over a common denominator then gives

$$\overline{OP} = \frac{(m+n)\mathbf{a} + m(\mathbf{b} - \mathbf{a})}{m+n}.$$

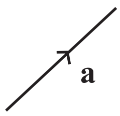
If we expand the brackets, the term $m\mathbf{a}$ will cancel with the term $m(-\mathbf{a})$, and so finally we have

$$\overline{OP} = \frac{n\mathbf{a} + m\mathbf{b}}{m+n}.$$

This formula gives us a way of calculating the position vector of the point P. For instance, if m and n were both 1 then P would be the mid-point of AB. The position vector of the midpoint would be $(\mathbf{a} + \mathbf{b})/2$. As another example, if $m = 2$ and $n = 1$, so that P was two-thirds of the way along the line, then the position vector of P would be $(\mathbf{a} + 2\mathbf{b})/3$.

Exercises

1. The vector $\mathbf{a}$ is shown below.

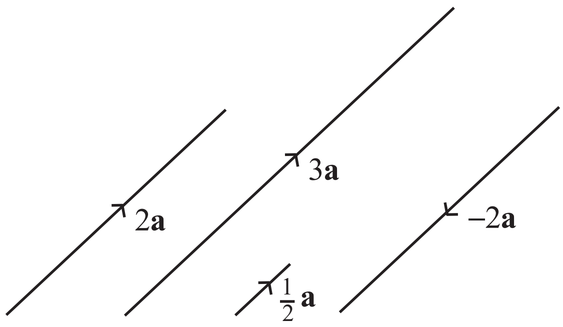


Sketch the vectors $2\mathbf{a}$, $3\mathbf{a}$, $\frac{1}{2}\mathbf{a}$ and $-2\mathbf{a}$.

2. In ΔOAB, $\overline{OA} = \mathbf{a}$ and $\overline{OB} = \mathbf{b}$. In terms of $\mathbf{a}$ and $\mathbf{b}$,
   (a) What is $\overline{AB}$?
   (b) What is $\overline{BA}$?
   (c) What is $\overline{OP}$, where P is the midpoint of AB?
   (d) What is $\overline{AP}$?
   (e) What is $\overline{BP}$?
   (f) What is $\overline{OQ}$, where Q divides AB in the ratio 2:3?
3. What is meant by a unit vector?
4. If $\mathbf{e}$ is a unit vector, what is the length of $3\mathbf{e}$?
5. In ΔABC, $AB = \mathbf{a}$, $BC = \mathbf{b}$, $CA = \mathbf{c}$. What is $\mathbf{a} + \mathbf{b} + \mathbf{c}$?

Answers

1.



2.
(a) $\mathbf{b} - \mathbf{a}$ (b) $\mathbf{a} - \mathbf{b}$ (c) $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ (d) $\frac{1}{2}(\mathbf{b} - \mathbf{a})$
(e) $\frac{1}{2}(\mathbf{a} - \mathbf{b})$ (f) $\frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$

3. A vector with length 1

4. 3

5. 0

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Prepared as part of the **math**centre community project by Anne Townsend with the assistance of Janette Matthews and Tony Croft. (October 8th, 2013)

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Arithmetic

When multiplying or dividing positive and negative numbers the sign of the result is given by:

positive × positive = positive positive × negative = negative
negative × positive = negative negative × negative = positive

$$\frac{\text{positive}}{\text{positive}} = \text{positive} \qquad \frac{\text{positive}}{\text{negative}} = \text{negative}$$

$$\frac{\text{negative}}{\text{positive}} = \text{negative} \qquad \frac{\text{negative}}{\text{negative}} = \text{positive}$$

The BIDMAS rule reminds us of the order in which operations are carried out. BIDMAS stands for:

Brackets ()	First priority
Indices ×	Second priority
Division ÷	Third priority
Multiplication ×	Third priority
Addition +	Fourth priority
Subtraction −	Fourth priority

Fractions

$$\text{fraction} = \frac{\text{numerator}}{\text{denominator}}$$

Adding and subtracting fractions. To add or subtract two fractions first rewrite each fraction so that they have the same denominator. Then, the numerators are added or subtracted as appropriate and the result is divided by the common denominator: e.g.

$$\frac{4}{5} + \frac{3}{4} = \frac{16}{20} + \frac{15}{20} = \frac{31}{20}$$

Multiplying fractions. To multiply two fractions, multiply their numerators and then multiply their denominators: e.g.

$$\frac{3}{7} \times \frac{5}{11} = \frac{15}{77}$$

Dividing fractions. To divide two fractions, invert the second and then multiply: e.g.

$$\frac{3}{5} \div \frac{2}{3} = \frac{3}{5} \times \frac{3}{2} = \frac{9}{10}$$

Decimals

Decimals are a type of fraction. Usually a fraction is written in the form $\frac{\text{numerator}}{\text{denominator}}$. Decimals are fractions in which the denominator is a power of 10, that is 10, 100, 1000 and so on, but instead of writing them in the usual form only the numerator is written down and a decimal point is used to indicate the size of the denominator.

Decimal fractions:

Look at the following fractions. n every case the denominator is a power of 10:

$$\frac{7}{10}, \quad \frac{5}{100}, \quad \frac{3}{1000}$$

n decimal form we would write

$$\frac{7}{10} = 0.7, \quad \frac{5}{100} = 0.05, \quad \frac{3}{1000} = 0.003$$

The frst position to the right of the decimal point indicates tenths. The second position indicates hundredths, the third indicates thousandths and so on.

A mixed number like $6\frac{3}{1}$ will consist of the whole number part on the left of the decimal point and the fractional part on the right, that is $6\frac{3}{1} = 6.3$.

Multiplying or dividing by powers of 10:

To multiply 36.57 by 10 digits are moved one place to the left,

$$36.57 \times 10 = 365.7$$

To multiply by 100 digits are moved two places to the left. So

$$78.375 \times 100 = 7837.5$$

Similarly

$$0.0095 \times 1000 = 9.5$$

To divide a number by 10 the digits are moved one place to the right. To divide by 100 the digits are moved two places to the right. For example

$$36.7 \div 10 = 3.67, \qquad 5.8 \div 10 = 0.58$$

$$0.0475 \div 100 = 0.000475$$

Converting a fraction to a decimal:

To convert a fraction into a decimal remember that $\frac{a}{b}$ means $a \div b$. Often a calculator can be used to perform the division.

Metric measures (cgs)

Length	Weight/Mass	Capacity
10 mm = 1 cm	1000 mg = 1 g	1 ml = 1000 mm^3
100 cm = 1 m	1000 g = 1 kg	10 ml = 1 cl
1000 m = 1 km	1000 kg = 1 tonne	100 cl = 1 litre
		1000 cm^3 = 1 litre

Imperial measures

Length	Weight/Mass	Capacity
12 inches = 1 foo t(ft)	16 ounces (oz)	20 luid oz
3 ft = 1 yard (yd)	= 1 pound (lb)	= 1 pint (pt)
1760 yds = 1 mile	14 lb = 1 stone	8 pt = 1 gallon

Time

60 seconds = 1 minute	52 weeks = 1 year
60 minutes = 1 hour	12 months = 1 year
24 hours = 1 day	10 years = 1 decade
7 days = 1 week	100 years = 1 century

Averages

Suppose we have a set of numbers. There are three common types of average:

Mean	$\frac{\text{Sum of the numbers}}{\text{number of items of data}}$
Median	middle number in an ordered set of data
Mode	number which occurs most often

Spread

The range tells us about how widely spread the data values are:

Range = highest value - lowest value

Interquartile range = upper quartile - lower quartile

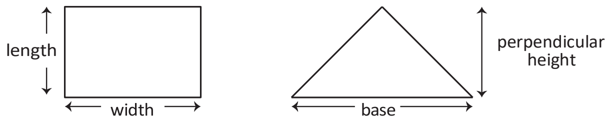
Probability

The probability of an event occurring is a number between 0 and 1.

The probability can be calculated from:

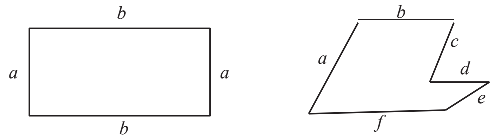
$$\frac{\text{number of outcomes for an event}}{\text{total number of outcomes}}$$

Area



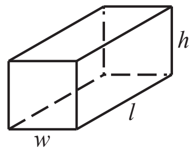
Rectangle	Triangle
Area = length x width	Area = $\frac{1}{2}$ x base x height
A = lw	A = $\frac{1}{2}$bh

Perimeter



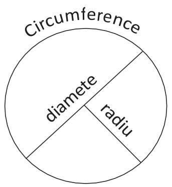
Rectangular shape	rregular shape
erimeter=sum of all lengths	erimeter=sum of all lengths
$= a + b + a + b$	$= a + b + c + d + e + f$

Volume



$$\begin{aligned} \text{Volume} &= \text{length} \times \text{width} \times \text{height (or depth)} \\ &= l \times w \times h \end{aligned}$$

Circles

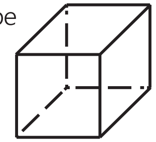


$$\begin{aligned} \text{Circumference} &= 2 \times \pi \times \text{radius} \\ &= 2\pi r \end{aligned}$$

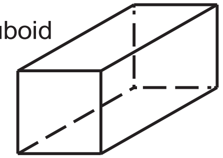
$$\begin{aligned} \text{Area} &= \pi \times \text{radius} \times \text{radius} \\ &= \pi r^2 \end{aligned}$$

3D Shapes

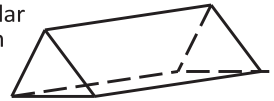
cube



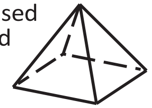
cuboid



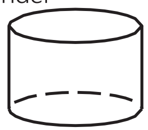
triangular prism



square-based pyramid



cylinder



30 shape	number of faces	number of edges	number of vertices
cube square prism		12	
cuboid rectangular prism		12	
cylinder circular prism	3	2	
square-based pyramid	5		5
triangular prism	5		